

Section 2: Completing the square

Exercise solutions

1.

i)

$$\begin{aligned}x^2 + 2x - 3 &= (x+1)^2 - 1 - 3 \\ &= (x+1)^2 - 4\end{aligned}$$

ii)

$$\begin{aligned}x^2 - 6x + 1 &= (x-3)^2 - 9 + 1 \\ &= (x-3)^2 - 8\end{aligned}$$

iii)

$$\begin{aligned}x^2 + x + 1 &= \left(x + \frac{1}{2}\right)^2 - \frac{1}{4} + 1 \\ &= \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}\end{aligned}$$

iv)

$$\begin{aligned}x^2 - 3x + 4 &= \left(x - \frac{3}{2}\right)^2 - \frac{9}{4} + 4 \\ &= \left(x - \frac{3}{2}\right)^2 - \frac{5}{4}\end{aligned}$$

2.

i)

$$\begin{aligned}3x^2 + 6x + 2 &= 3(x^2 + 2x) + 2 \\ &= 3((x+1)^2 - 1) + 2 \\ &= 3(x+1)^2 - 3 + 2 \\ &= 3(x+1)^2 - 1\end{aligned}$$

ii)

$$\begin{aligned}-x^2 + 5x &= -(x^2 - 5x) \\ &= -\left(\left(x - \frac{5}{2}\right)^2 - \frac{25}{4}\right) \\ &= -\left(x - \frac{5}{2}\right)^2 + \frac{25}{4}\end{aligned}$$

iii)

$$\begin{aligned}
 2x^2 + 4x + 3 &= 2(x^2 + 2x) + 3 \\
 &= 2((x+1)^2 - 1) + 3 \\
 &= 2(x+1)^2 - 2 + 3 \\
 &= 2(x+1)^2 + 1
 \end{aligned}$$

iv)

$$\begin{aligned}
 3x^2 + 8x - 2 &= 3\left(x^2 + \frac{8}{3}x\right) - 2 \\
 &= 3\left(\left(x + \frac{4}{3}\right)^2 - \frac{16}{9}\right) - 2 \\
 &= 3\left(x + \frac{4}{3}\right)^2 - \frac{16}{3} - 2 \\
 &= 3\left(x + \frac{4}{3}\right)^2 - \frac{22}{3}
 \end{aligned}$$

3. Method 1- Completing the square

$$\begin{aligned}
 p - q(x+r)^2 &= 6 - 12x - 3x^2 \\
 &= -3(x^2 + 4x) + 6 \\
 &= -3((x+2)^2 - 4) + 6 \\
 &= -3(x+2)^2 + 12 + 6 \\
 &= 18 - 3(x+2)^2
 \end{aligned}$$

So, $p = 18$, $q = 3$ and $r = 2$ Method 2- Equating coefficients

$$\begin{aligned}
 p - q(x+r)^2 &= p - q(x^2 + 2xr + r^2) \\
 &= p - (qx^2 + 2qxr + qr^2) \\
 &= p - qr^2 - 2qxr - qx^2 \\
 &= 6 - 12x - 3x^2
 \end{aligned}$$

Equating coefficients:

$$p - qr^2 = 6$$

$$-2qxr = -12x, \text{ so } -2qr = -12$$

$$-qx^2 = -3x^2, \text{ so } -q = -3$$

From this we can deduce that: $q = 3$ Substituting this value for q into the above equations:

$$-2(3)r = -12$$

$$-6r = -12$$

$$\text{So, } r=2$$

$$p - qr^2 = 6$$

$$p - (3)(2)^2 = 6$$

$$p - 12 = 6$$

$$p = 18$$

$$\text{So, } p = 18, q = 3 \text{ and } r = 2$$

4. Method 1- Completing the square

$$\begin{aligned} c - a(x-2)^2 &= 8 + bx - 4x^2 \\ &= -4\left(x^2 - \frac{b}{4}x\right) + 8 \\ &= -4\left(\left(x - \frac{b}{8}\right)^2 - \frac{b^2}{64}\right) + 8 \\ &= -4\left(x - \frac{b}{8}\right)^2 + \frac{b^2}{16} + 8 \\ &= \frac{128 + b^2}{16} - 4\left(x - \frac{b}{8}\right)^2 \end{aligned}$$

We can see from this $a = 4$

$$\text{Also, } \frac{b}{8} = 2 \text{ so } b = 16$$

$$\text{So, } c = \frac{128 + b^2}{16} = \frac{128 + 256}{16} = 24$$

Method 2- Equating coefficients

$$\begin{aligned} c - a(x-2)^2 &= c - a(x^2 - 4x + 4) \\ &= c - ax^2 + 4ax - 4a \\ &= c - 4a + 4ax - ax^2 \\ &= 8 + bx - 4x^2 \end{aligned}$$

Equating coefficients:

$$c - 4a = 8$$

$$bx = 4ax \text{ so } b = 4a$$

$$-4x^2 = -ax^2 \text{ so } a = 4$$

$$\text{So, } b = 4(4) = 16$$

$$c = 8 + 4a = 8 + 4(4) = 8 + 16 = 24$$

5.

$$\begin{aligned}x^2 - 4x + 8 &= (x-2)^2 - 4 + 8 \\&= (x-2)^2 + 4\end{aligned}$$

OR

$$\begin{aligned}(x-2)^2 + 4 &= (x^2 - 4x + 4) + 4 \\&= x^2 - 4x + 4\end{aligned}$$

Making x the subject of the formula:

$$y = x^2 - 4x + 8$$

$$\text{So, } y = (x-2)^2 + 4$$

$$(x-2)^2 = y - 4$$

$$x - 2 = \pm \sqrt{y - 4}$$

$$x = 2 \pm \sqrt{y - 4}$$

6. First, complete the square:

$$y = 3x^2 + 8x - 3$$

$$= 3\left(x^2 + \frac{8}{3}x\right) - 3$$

$$= 3\left(\left(x + \frac{4}{3}\right)^2 - \frac{16}{9}\right) - 3$$

$$= 3\left(x + \frac{4}{3}\right)^2 - \frac{16}{3} - 3$$

$$= 3\left(x + \frac{4}{3}\right)^2 - \frac{25}{3}$$

Then rearrange to make x the subject of the formula:

$$3\left(x + \frac{4}{3}\right)^2 - \frac{25}{3} = y$$

$$3\left(x + \frac{4}{3}\right)^2 = y + \frac{25}{3}$$

$$\left(x + \frac{4}{3}\right)^2 = \frac{y}{3} + \frac{25}{9} = \frac{3y + 25}{9}$$

$$x + \frac{4}{3} = \pm \sqrt{\frac{3y + 25}{9}}$$

$$x = \pm \frac{\sqrt{3y + 25}}{3} - \frac{4}{3} = \pm \frac{\sqrt{3y + 25} - 4}{3}$$