

Section 1: Factorising, algebraic fractions and formulae

Notes and Examples

These notes contain subsections on

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Factorising

To factorise an expression, look for numbers and/or letters which are common factors of each term. We often talk about “taking out a factor” – this can cause confusion as it tends to make you think that subtraction is involved. In fact you are, of course, dividing each term by the common factor which you are “taking out”.

You can check your answers by multiplying out the brackets.



Example 1

Factorise the following expressions.

- $6a + 12b + 3c$
- $6x^2y - 10xy^2 + 2xy$
- $3(x + y) - 2x(x + y)$



Solution

- 3 is a factor of each term.

$$6a + 12b + 3c = 3(2a + 4b + c)$$
- $2xy$ is a factor of each term.

$$6x^2y - 10xy^2 + 2xy = 2xy(3x - 5y + 1)$$
- $x + y$ is a factor of each term.

$$3(x + y) - 2x(x + y) = (x + y)(3 - 2x)$$

Factorising quadratic expressions

When you multiply out an expression involving two brackets, like $(2x - 1)(x + 3)$, you get a **quadratic expression**, which involves a term in x^2 .

$$\begin{aligned}(2x - 1)(x + 3) &= 2x^2 - x + 6x - 3 \\ &= 2x^2 + 5x - 3\end{aligned}$$

Factorising a quadratic expression is the reverse of this process. Example 2 shows how you can do this.



Example 2

Factorise the expressions

- (i) $x^2 + 4x + 3$
- (ii) $x^2 - 4x - 12$
- (iii) $2x^2 - 7x + 6$

Solution

(i) $x^2 + 4x + 3 = (x \dots)(x \dots)$

Start with an x in each bracket

$$x^2 + 4x + 3 = (x + 1)(x + 3)$$

You need two numbers whose sum is 4 and whose product is 3. These are +1 and +3.

(ii) $x^2 - 4x - 12 = (x \dots)(x \dots)$

Start with an x in each bracket

$$x^2 - 4x - 12 = (x - 6)(x + 2)$$

You need two numbers whose sum is -4 and whose product is -12. These are -6 and +2.

(iii) $2x^2 - 7x + 6 = (2x \dots)(x \dots)$

In this case you need to start with $2x$ in one bracket and x in the other.

$$2x^2 - 7x + 6 = (2x - 3)(x - 2)$$

It is not so straightforward to find the two numbers in this case, because of the $2x$ in one bracket. The two numbers must have a product of +6, and as the coefficient of x is negative, they must both be negative. Try the different possibilities (-1 and -6, or -2 and -3, in either order), until you find the correct one.

Expressions which involve terms in x^2 , xy and y^2 can be factorised in a similar way – each bracket involves an x term and a y term.



Example 3

Factorise

- (i) $x^2 + 2xy - 3y^2$

(ii) $2x^2 - 11xy + 12y^2$

Solution

(i) $x^2 + 2xy - 3y^2 = (x + 3y)(x - y)$

(ii) $2x^2 - 11xy + 12y^2 = (2x - 3y)(x - 4y)$



The difference of two squares

One important special case of a quadratic expression that you should recognise is called 'the difference of two squares'.

An example is the expression $x^2 - 4$. Both x^2 and 4 are squares, since 4 can be written as 2^2 . This is a quadratic expression with no 'middle term'.

The expression $x^2 - 4$ can be factorised to give $(x + 2)(x - 2)$. Check this by multiplying out.

More generally, any expression of the form $a^2 - b^2$ can be written as $(a + b)(a - b)$. Look out for 'difference of two squares' expressions. The example below shows how using this result can save some work.

Example 4

Factorise fully $(3x + 1)^2 - (2x - 3)^2$.

Notice that this expression is the difference between two squares.

Solution

$$\begin{aligned} (3x + 1)^2 - (2x - 3)^2 &= ((3x + 1) + (2x - 3))((3x + 1) - (2x - 3)) \\ &= (3x + 1 + 2x - 3)(3x + 1 - 2x + 3) \\ &= (5x - 2)(x + 4) \end{aligned}$$

We can use $a^2 - b^2 = (a + b)(a - b)$, with a as $3x + 1$ and b as $2x - 3$.

In Example 4, you could multiply out both brackets, simplify and then factorise – this would involve a lot more work than the solution shown above!

You can see step-by-step examples of factorising quadratics in this [PowerPoint presentation](#).

You can also look at the [Factorising quadratics video](#).

For practice in examples like the ones above, try the interactive questions [Factorising quadratics](#).

You can also test yourself using the Flash resource [Factorising quadratics](#).

Changing the subject of a formula



Changing the subject of a formula is similar to solving an equation, but you are working with letters rather than numbers.

Example 5

The volume V of a cylinder with radius r and height h is given by $V = \pi r^2 h$.
Make r the subject of this formula.

Solution

$$V = \pi r^2 h$$

$$\frac{V}{\pi h} = r^2$$

$$\sqrt{\frac{V}{\pi h}} = r$$

$$r = \sqrt{\frac{V}{\pi h}}$$

Divide both sides by πh

Square root both sides

Finish by writing the equation with r on the left side.

In the next example, two different methods are shown. The answers look a bit different but they are equivalent. Make sure that you can see how you could rewrite one solution to give the other.

Example 6

The surface area A of a cylinder with radius r and height h is given by $A = 2\pi r(h + r)$.
Make h the subject of this formula.

Solution (1)

$$A = 2\pi r(h + r)$$

$$A = 2\pi rh + 2\pi r^2$$

$$A - 2\pi r^2 = 2\pi rh$$

$$\frac{A - 2\pi r^2}{2\pi r} = h$$

$$h = \frac{A - 2\pi r^2}{2\pi r}$$

Multiply out the brackets

Subtract $2\pi r^2$ from each side

Divide each side by $2\pi r$

Rewrite with h on the left

Solution (2)

$$A = 2\pi r(h + r)$$

$$\frac{A}{2\pi r} = h + r$$

$$\frac{A}{2\pi r} - r = h$$

$$h = \frac{A}{2\pi r} - r$$

Divide each side by $2\pi r$

Subtract r from each side

Rewrite with h on the left side

In the next example, the new subject appears more than once. You need to collect the terms involving the new subject together and then factorise to isolate the new subject.



Example 7

Make x the subject of the formula $cx - a = a(b + x)$.

Solution

$$cx - a = a(b + x)$$

$$cx - a = ab + ax$$

$$cx - ax - a = ab$$

$$cx - ax = ab + a$$

$$(c - a)x = a(b + 1)$$

$$x = \frac{a(b + 1)}{c - a}$$

Multiply out the brackets

Subtract ax from each side to collect the terms in x together

Add a to each side

Factorise

Divide both sides by $c - a$

Algebraic fractions

Algebraic fractions follow the same rules as numerical fractions. When adding or subtracting, you need to find the common denominator, which may be a number or an algebraic expression.



Example 8

Write as single fractions:

(i) $\frac{2x}{3} + \frac{x}{4} - \frac{5x}{6}$

(ii) $\frac{1}{2x} - \frac{1}{x^2}$

(iii) $\frac{2}{x-1} + \frac{3}{x+2}$

Solution

(i) The common denominator is 12, as 3, 4 and 6 are all factors of 12.

$$\begin{aligned} \frac{2x}{3} + \frac{x}{4} - \frac{5x}{6} &= \frac{8x}{12} + \frac{3x}{12} - \frac{10x}{12} \\ &= \frac{8x + 3x - 10x}{12} \\ &= \frac{x}{12} \end{aligned}$$



- (ii) The common denominator is $2x^2$.

$$\begin{aligned}\frac{1}{2x} - \frac{1}{x^2} &= \frac{x}{2x^2} - \frac{2}{2x^2} \\ &= \frac{x-2}{2x^2}\end{aligned}$$

- (iii) The common denominator is $(x-1)(x+2)$.

$$\begin{aligned}\frac{2}{x-1} + \frac{3}{x+2} &= \frac{2(x+2)}{(x-1)(x+2)} + \frac{3(x-1)}{(x+2)(x-1)} \\ &= \frac{2x+4+3x-3}{(x-1)(x+2)} \\ &= \frac{5x+1}{(x-1)(x+2)}\end{aligned}$$

You are familiar with the idea of “cancelling” to simplify numerical fractions: for example, $\frac{9}{12}$ can be simplified to $\frac{3}{4}$ by dividing both the numerator and the denominator by 3. You can also cancel before carrying out a multiplication, to make the numbers simpler:

e.g. $\frac{2}{3} \times \frac{3}{2} = 1$. The same technique can be used in algebra. As with factorising, remember that “cancelling” involves dividing, not subtracting.

Example 9

Simplify

(i) $\frac{6xy^3 + 2x^2y}{10x^2y}$

(ii) $\frac{3a}{a+1} \times \frac{2a+2}{a+2}$

Solution

(i)
$$\begin{aligned}\frac{6xy^3 + 2x^2y}{10x^2y} &= \frac{2xy(3y^2 + x)}{10x^2y} \\ &= \frac{3y^2 + x}{5x}\end{aligned}$$

It is very important to remember that you can only “cancel” if you can divide each term in both the numerator and denominator by the same expression. In this case, don’t be tempted to divide by $2x^2y$ – although this is a factor of both $2x^2y$ and $10x^2y$, it is not a factor of $6xy^3$. In a case like this, it may be best to factorise the top first, so that it is easier to see the factors.

$2xy$ is a common factor of both top and bottom

- (ii) Again, factorise where possible first.

$$\begin{aligned}\frac{3a}{a+1} \times \frac{2a+2}{a+2} &= \frac{3a}{a+1} \times \frac{2(a+1)}{a+2} \\ &= \frac{6a}{a+2}\end{aligned}$$

$(a+1)$ is a common factor of both top and

Notice that you cannot cancel a here, as it is not a factor of $a+2$.

Sometimes algebraic expressions which look quite complicated can be simplified by factorising.

**Example 10**

Simplify $\frac{x^2 - 1}{x^2 + 2x - 3}$.

It can be tempting to try to "cancel" x^2 from the top and the bottom. Don't!
You can only cancel something which is a factor of the top and the bottom.

Solution

$$\begin{aligned}\frac{x^2 - 1}{x^2 + 2x - 3} &= \frac{(x+1)(x-1)}{(x+3)(x-1)} \\ &= \frac{x+1}{x+3}\end{aligned}$$

You can now cancel the factor $(x - 1)$ from the top and the bottom.

