AQA Level 2 Further Mathematics Algebra II

Section 2: Completing the square

Notes and Examples

These notes contain subsections on

- **Completing the square**

Completing the square

Sometimes it is useful to write a quadratic expression in the form $a(x + b)^2 + c$. This is called the completed square form. This form can be useful to help you find out more about a quadratic expression. For example, the expression $2(x – 1)^2 + 3$ has a minimum value of 3, since the squared term cannot be less than zero. This wouldn't be immediately obvious if the expression were given in its expanded form of $2x^2 – 4x + 5$.

The examples below demonstrate the technique of completing the square.

Example 1

Write the expression $x^2 + 4x + 7$ in the form $(x + b)^2 + c$.

> In this case you are asked for $(x + b)^2 + c$ rather than $a(x + b)^2 + c$. This is because the coefficient of x^2 is 1.

Solution

> First you need to find a quadratic expression which is a perfect square and which begins with $x^2 + 4x$.
> You do this by looking at the coefficient of x, in this case 4, and halving it. In this case you get 2.
> This tells you that the perfect square you need is $(x + 2)^2$.

$(x + 2)^2 = x^2 + 4x + 4$

$x^2 + 4x + 7 = x^2 + 4x + 4 + 3$

$\qquad = (x + 2)^2 + 3$

> The +3 'completes the square'

> This is why the technique is called 'completing the square'.

> $(x + 2)^2$ is the 'square'

Alternative Solution (equating coefficients)
Expanding $(x + b)^2 + c$ gives $x^2 + 2bx + b^2 + c$.

Equating coefficients between this and $x^2 + 4x + 7$ gives

$2b = 4$ and $b2 + c = 7$.

From the first equation $b = 2$. Substituting $b = 2$ in the second equation gives $4 + c = 7$ and so $c = 3$.

Therefore $x^2 + 4x + 7 = (x + 2)^2 + 3$.

As you can see, there are several different approaches to writing out the working. They are all basically the same, so if you have learnt a different way which suits you, then stick to it.

The next example shows a situation where the coefficient of x^2 is not 1.

Example 12
Write the expression $2x^2 - 6x + 1$ in the form $a(x + b)^2 + c$.

Solution

Start by taking out the coefficient of x^2, in this case 2, as a factor.

$2x^2 - 6x + 1 = 2(x^2 - 3x + \frac{1}{2})$

Now look at the expression inside the bracket. You need to find a quadratic expression which is a perfect square and starts with $x^2 - 3x$. Take the coefficient of x, which is –3, and halve it to get $-\frac{3}{2}$. The perfect square you need is therefore $\left(x - \frac{3}{2}\right)^2$.

$\left(x - \frac{3}{2}\right)^2 = x^2 - 3x + \frac{9}{4}$

$x^2 - 3x + \frac{1}{2} = x^2 - 3x + \frac{9}{4} - \frac{9}{4} + \frac{1}{2}$
$= (x - \frac{3}{2})^2 - \frac{9}{4} + \frac{1}{2}$
$= (x - \frac{3}{2})^2 - \frac{7}{4}$
$2x^2 - 6x + 1 = 2[(x - \frac{3}{2})^2 - \frac{7}{4}]$
$= 2(x - \frac{3}{2})^2 - \frac{7}{2}$

In the next example, the coefficient of x^2 is negative. This can be dealt with by taking out a factor -1.

01/08/18 © MEI

Example 14

Write the expression $5 + x - x^2$ in the form $p - q(x - r)^2$.

Solution

Start by taking out -1 as a factor.

$$5 + x - x^2 = -(x^2 - x - 5)$$

Now you need a quadratic expression which is a perfect square and starts with $x^2 - x$. Half the coefficient of x is $-\frac{1}{2}$, so the perfect square you need is $(x - \frac{1}{2})^2$.

$$(x - \tfrac{1}{2})^2 = x^2 - x + \tfrac{1}{4}$$

$$x^2 - x - 5 = x^2 - x + \tfrac{1}{4} - \tfrac{1}{4} - 5$$
$$= (x - \tfrac{1}{2})^2 - \tfrac{1}{4} - 5$$
$$= (x - \tfrac{1}{2})^2 - \tfrac{21}{4}$$

$$5 + x - x^2 = -[(x - \tfrac{1}{2})^2 - \tfrac{21}{4}]$$
$$= \tfrac{21}{4} - (x - \tfrac{1}{2})^2$$