AQA Level 2 Further Mathematics Geometry II

Section 2: Lines and planes in 3D

Notes and Examples

- **Working in three dimensions**
- **Angles between lines and planes**
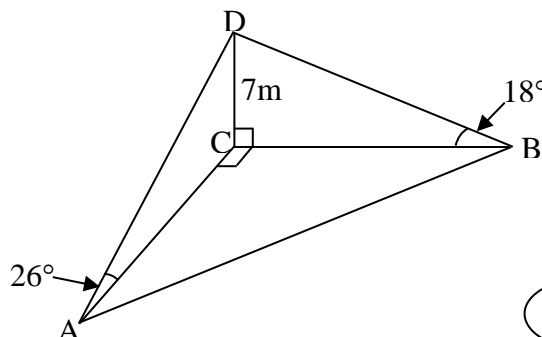
Working in three dimensions

To deal with three-dimensional problems, you will usually need to draw several diagrams. You need to identify two-dimensional triangles that will help you to find the length or angle you need. Label the vertices on your diagrams carefully so that you can see how they relate to the three-dimensional diagram.
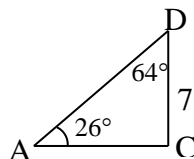
Example 1
A man standing at A observes a tower CD due north of him. The tower is 7m in height and has an angle of elevation of 26° from A. Point B is due east of the tower and the angle of elevation here is 18°. A, B and C all lie in the same horizontal plane. Find the distance from A to B and the bearing of B from A.
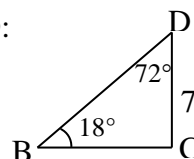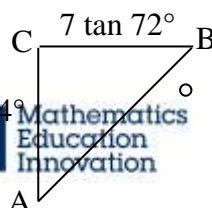
Solution

From triangle ACD we can see that the angle at D is 64°. Use this to avoid having to divide by tan 26°

Using triangle ACD:

AC $= 7 \tan 64°$
$\quad = 14.4\text{m}$

Similarly with triangle BCD:

BC $= 7 \tan 72°$
$\quad = 21.5\text{m}$

From triangle ABC:

Use the exact values for the lengths in your calculations

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$$AB = AC^2 + BC^2$$
$$= 25.9\text{m}$$

$$\tan A = \frac{BC}{AC}$$

Angle A = 56.2° and hence the bearing of B from A is 056°.

Notice that you should always use exact values in your calculations, to avoid rounding errors. Write lengths on your diagrams in exact form, using trig or square root signs, as shown in the example above and in the next one.
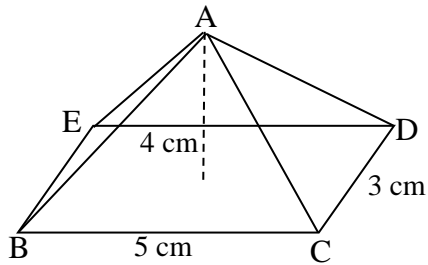
Angles between lines and planes

You may be asked to find the angle between a line and a plane, or the angle between two planes. You need to think carefully about which triangle to use.

- For an angle between a line and a plane, imagine the line casting a shadow on the plane. The position of the shadow is the line you need to use in your triangle. Hold a pencil at an angle to a flat surface, and make sure that you can identify the angle you would need.

- For an angle between two planes, you need to use two lines which are at right-angles to the line joining the planes – this gives the steepest slope possible. Hold a piece of paper at an angle to a flat surface, and make sure that you can identify the angle you would need.
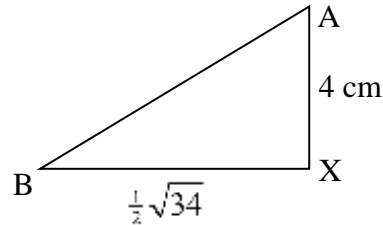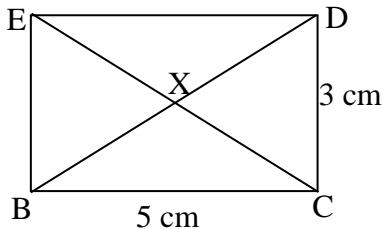
Example 2
The diagram shows a pyramid with a rectangular base with length 5 cm and width 3 cm. The vertex A of the pyramid is directly above the centre of the rectangle BCDE, at height 4 cm.

(i)   Find the length AB.
(ii)   Find the angle between the line AB and the plane BCDE.
(iii) Find the angle between the plane ABC and the plane BCDE.

Solution

(i)   To find the length AB, use the triangle ABX, where X is the centre of the rectangle BCDE.

$$BD^2 = 3^2 + 5^2 = 34$$

$$BD = \sqrt{34}$$

$$BX = \tfrac{1}{2}\sqrt{34}$$

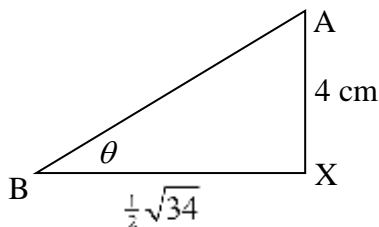$$AB^2 = 4^2 + \left(\tfrac{1}{2}\sqrt{34}\right)^2$$

$$= 16 + \tfrac{1}{4} \times 34$$

$$= 24.5$$

$$AB = \sqrt{24.5} = 4.95 \ (3 \text{ s.f.})$$

So AB = 4.95 cm (3 s.f.)

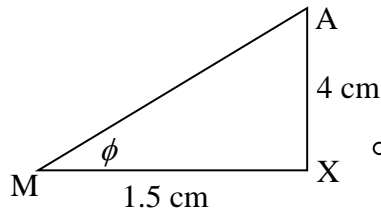(ii)   To find the angle between the line AB and the plane BCDE, use the triangle ABX again.

$$\tan\theta = \frac{4}{\tfrac{1}{2}\sqrt{34}} = \frac{8}{\sqrt{34}}$$

$$\theta = 53.9 \ (3 \text{ s.f.})$$

The angle between AB and the plane is 53.9° (3 s.f.).

(iii)   To find the angle between the plane ABC and the plane BCDE, use the triangle AMX, where M is the midpoint of BC.

> MX is parallel to CD and is half its length, so MX = 1.5 cm

$$\tan \phi = \frac{4}{1.5} = \frac{8}{3}$$

$\phi = 69.4$ (3 s.f.)

So the angle between the two planes is 69.4° (3 s.f.)

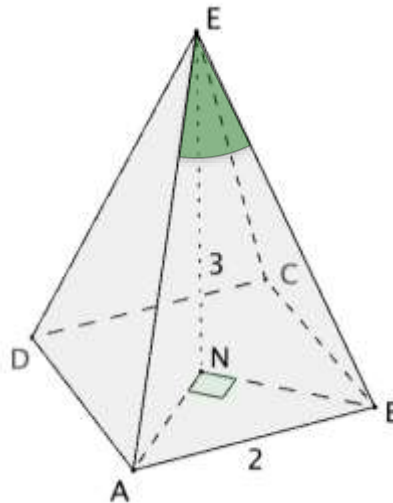**Example 3**

A pyramid has a square base, ABCD with side length 2. The apex E is 3 units above the centre of the base. Find $\text{Đ}AEB$.

Solution

Draw a diagram

> Sometimes the information given doesn't lead directly to the answer. You may have to work through several different stages

AE = AB
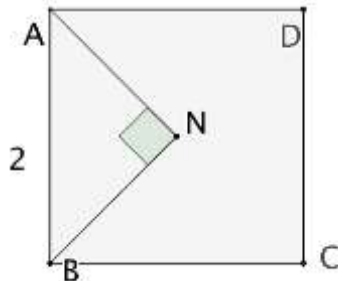
The length AE needs to be found.

Start by finding AN:

$AN^2 + BN^2 = 2^2$

Since $AN = BN$

$2AN^2 = 4$

$AN^2 = 2$

$AN = \sqrt{2}$

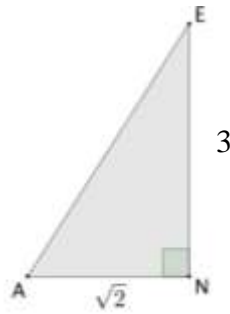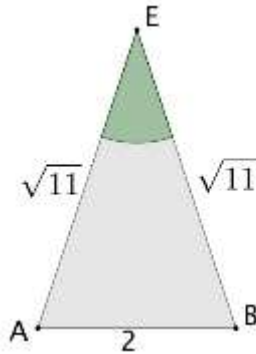Using this result find AE:

$$\left(\sqrt{2}\right)^2 + 3^2 = AE^2$$

$$2 + 9 = AE^2$$

$$AE = \sqrt{11}$$

3

Now we look at triangle AEB

$\sqrt{11}$ $\sqrt{11}$

A 2 B

Keep the answers to each stage as accurate as possible. Calculate and round only at the very end

$$\cos(\angle AEB) = \frac{a^2 + b^2 - e^2}{2ab}$$

$$\cos(\angle AEB) = \frac{\sqrt{11}^2 + \sqrt{11}^2 - 2^2}{2\sqrt{11}\sqrt{11}} = \frac{11 + 11 - 4}{22}$$

$$= \frac{18}{22} = \frac{9}{11}$$

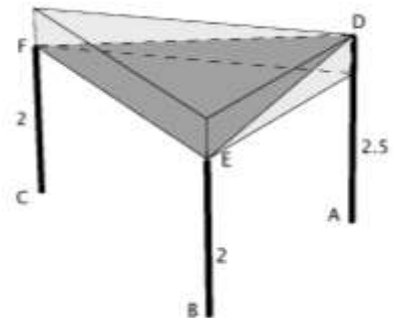$$\angle AEB = \cos^{-1}\left(\frac{9}{11}\right) = 35.1°$$

You may use several rules in one question. This example uses Pythagoras' Theorem and the cosine rule
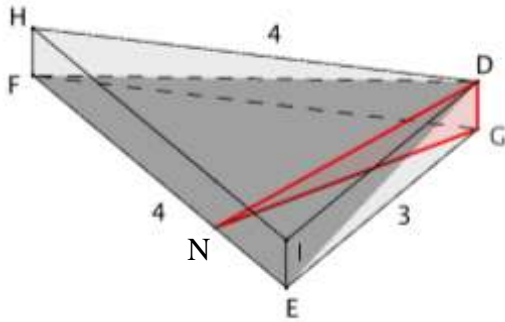
Example 4

Amanda is trying to build a sunshade in her garden. The sunshade is a piece of triangular fabric stretched between the tops of three poles. She has 2 poles which are 2m tall and one of which is 2.5m. She puts the 2.5m pole in the ground at point A and the 2m poles at point B and C. AB = 3m, AC = BC = 4m. To ensure that the rain runs off the shade the angle of greatest slope must be more than 10°.

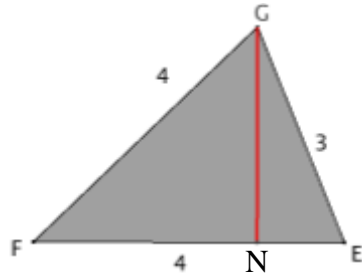Determine whether this setup will ensure an angle of greatest slope of more than 10°.

Solution

In some setups it may be difficult to see where the angle of greatest slope would be. In this case it Ð*GND* because DN and GN are perpendicular to EF.

N is on EF and is located such that GN is perpendicular to EF.

Plan view:

It is often useful to redraw parts of the diagram at each stage with just the relevant information for that stage.
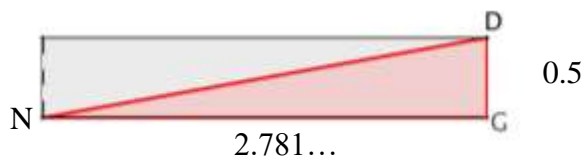
$$\cos(\angle GFE) = \frac{4^2 + 4^2 - 3^2}{2*4*4}$$

$$\cos(\angle GFE) = \frac{23}{32}$$

$$\angle GFE = \cos^{-1}\left(\frac{23}{32}\right) = 44.0486...$$

Do not round your answer part way through a question. Use your calculator to save the exact number and use that in later parts.

$$GN = 4\sin(\text{Ð}GFE) = 2.781...$$

Side view:

$$\tan(\text{Ð}DNG) = \frac{0.5}{2.781...}$$

$$\text{Ð}DNG = 10.2$$

The angle of greatest slope is just greater than 10° so the rain will run off.