Section 2: The binomial expansion

Notes and Examples
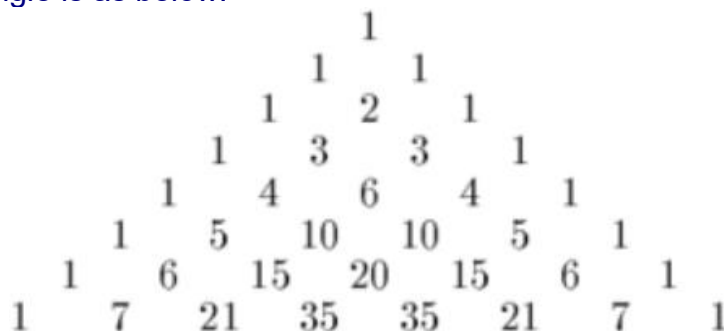
These notes contain subsections on:
- **Using Pascal's triangle**
- **Combinations**
- **The formula for binomial coefficients**
- **Binomial expansion using the formula**

In this section you will learn to multiply out expressions of the form $(a+b)^n$, where n is a positive integer.

Using Pascal's triangle

Pascal's triangle is as below.

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \\
\end{array}
$$

The entries at the start and end of each row are 1 and any other entry is the sum of the two entries immediately above it. In the below the first row (the single '1' should be thought of as the row zero, so that '1   1' is row n.

Expanding an expression of the form $(a+b)^n$ using Pascal's triangle is quite straightforward.

- Find the appropriate row of binomial coefficients from Pascal's triangle (the first two numbers in the row are 1 and n, and the row has $n+1$ numbers in it in total)
- The first term is a^n and the powers of a go down by 1 in each term, so that the last term has no a in it.
- The first term has no b in it and the powers of b go up by 1 in each term, so that the last term is b^n.

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Example 1

Expand $(p+q)^5$.

Solution

> The row you need from Pascal's triangle starts with 1, 5 and has 6 numbers in it. The row is 1, 5, 10, 10, 5, 1.

$$(p+q)^5 = p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5$$

You expand $(a+b)^n$ in the same way whether a and b are numbers or letters. If they are numbers a little more work is involved as you need to work out the actual powers and multiply by the binomial coefficients. This is shown in the next example.

Example 2

Expand $(x+2)^6$

Solution

Using Pascal's triangle, the binomial coefficients are 1, 6, 15, 20, 15, 6, 1.

$$(x+2)^6 = 1x^6 + 6x^5(2) + 15x^4(2)^2 + 20x^3(2)^3 + 15x^2(2)^4 + 6x(2)^5 + 1(2^6)$$
$$= x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64$$

If a negative is involved, remember that even powers of a negative number are positive, e.g. $(-2)^4 = 16$, but odd powers of a negative number are negative, e.g. $(-2)^3 = -8$.

Example 3

Expand $(2x-3y)^4$

Solution

From Pascal's triangle, the binomial coefficients are 1, 4, 6, 4, 1

$$(2x-3y)^4 = (2x)^4 + 4(2x)^3(-3y) + 6(2x)^2(-3y)^2 + 4(2x)(-3y)^3 + (-3y)^4$$
$$= 16x^4 + 4(8x^3)(-3y) + 6(4x^2)(9y^2) + 4(2x)(-27y^3) + 81y^4$$
$$= 16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$$

You can always expand brackets like the above using Pascal's triangle. An alternative method is presented in the rest of these

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notes.

Combinations

For large values of n, using Pascal's triangle can become complicated as you would need to write out a lot of rows.

It is useful to think about what is happening when you are multiplying out. Think about a fairly simple example: $(a+b)^4$.
In full, this is $(a+b)(a+b)(a+b)(a+b)$.
When you multiply out brackets like this, each term includes one term from each bracket. So, for example, $aaab$, $aaba$, $abaa$ and $baaa$ are four of the terms. Each of them can be written as a^3b, so these four terms can be written as $4a^3b$.

For a more complicated example, such as $(a+b)^{20}$, suppose you want to find the coefficient of a^5b^{15}. So you need to know how many times the term a^5b^{15} appears in the expansion. The question you need to ask is 'how many ways can I choose a from 5 of the 20 brackets? (For the brackets where you don't choose a, you will have to choose b, so that will give the b^{15}).

It's easier to think about this using a real-life context.

Example 4
Out of a class of 28 students, 3 students are to be chosen for the School Council. How many ways can the students be chosen?

Solution
There are 28 ways of picking the first student.
Once that student has been chosen, there are 27 ways of choosing the second, and then 26 ways of choosing the third.

This gives $28 \times 27 \times 26 = 19656$.

However, this gives too many combinations. If Alice were chosen first, Brendan second and Chloe third, this would be the same as if Chloe were chosen first, Alice second and Brendan third. In fact, there are 6 possible ways that Alice, Brendan and Chloe could be chosen:

      ABC ACB   BAC BCA CAB CBA

This is the case for any combination of three students. So all of the 19656 ways of choosing can be divided up into groups of 6 for which the 6 ways are all the same. So to the number of distinct combinations is $\dfrac{28 \times 27 \times 26}{6} = 3276$

To generalise the problem above, you need to think about where the 6 comes from. You needed to know how many ways Alice, Brendan and Chloe could

be chosen – in other words, in how many different orders can you write A, B and C?

You have three choices for the first letter, two for the second and one for the third. So the number of ways is $3 \times 2 \times 1 = 6$, as expected.

A quick notation for writing $3 \times 2 \times 1$ is to write $3!$ This is called **factorial** notation. In general, $n! = n \times (n-1) \times (n-2) \times ... \times 2 \times 1$.

So the calculation for choosing 3 people from 28 can be written as

$$\frac{28 \times 27 \times 26}{1 \times 2 \times 3}$$

This is equivalent to $\dfrac{28 \times 27 \times 26 \times (25 \times 24 \times ... \times 2 \times 1)}{1 \times 2 \times 3 \times (25 \times 24 \times ... \times 2 \times 1)}$

which can be written as $\dfrac{28!}{3! \, 25!}$.

You can generalise this to say that the number of ways of choosing r objects from n objects is given by $\dfrac{n!}{r!(n-r)!}$. This is written as $_nC_r$.

The formula for binomial coefficients

You can now go back to the problem of expanding brackets.

If you are expanding $(a+b)^n$, and you want to find the term that involves $a^r b^{n-r}$, then you need to choose the 'a' from r of the brackets and the 'b' from the rest of the brackets. So the coefficient of this term is the number of ways you can choose r out of the n brackets. This is the binomial coefficient and is given by $_nC_r$.

As shown above, the definition of the binomial coefficient $_nC_r$, or $\begin{pmatrix} n \\ r \end{pmatrix}$, is

$$\frac{n!}{r!(n-r)!}.$$

However, it is usually easier to use the form $\dfrac{n(n-1)....(n-r+1)}{1 \times 2 \times 3 \times ... \times r}$.
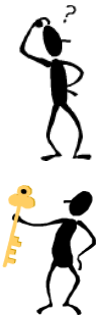
This looks complicated in algebra but it is much easier when working with numbers. The denominator is the product of the integers from 1 up to r, and the numerator is the product of the integers from n down until you have r numbers altogether. So there must be an equal number of integers in the numerator and the denominator.

If you have to work this out without a calculator, there will be a lot of cancelling you can do, so the calculation usually ends up quite simple. The example below demonstrates working out without a calculator.

Note that the binomial coefficients are symmetrical, so that $_nC_r = {_nC_{n-r}}$. So if you want to work out $_{15}C_{12}$, for example, you can work out $_{15}C_3$ instead, which is much easier.

Many calculators have a button for $_nC_r$, but you should practise working these out without a calculator, as shown in the example below, to familiarise yourself with the process.

Example 5
Find   (i) $_6C_3$ (ii) $_{10}C_8$ (iii) $_{12}C_4$

Solution

(i) $_6C_3 = \dfrac{\cancel{6} \times 5 \times 4}{1 \times \cancel{2} \times \cancel{3}} = 20$

> By symmetry $_{10}C_8 = {_{10}C_2}$

(ii) $_{10}C_8 = {_{10}C_2} = \dfrac{^5\cancel{10} \times 9}{1 \times \cancel{2}} = 45$

> 3 and 4 cancel with 12, and 2 cancels with 10 leaving 5.

(iii) $^{12}C_4 = {_{12}C_4} = \dfrac{\cancel{12} \times 11 \times {}^5\cancel{10} \times 9}{1 \times \cancel{2} \times 3 \times 4}$

$= 11 \times 5 \times 9$

$= 495$

Binomial expansion using the formula

Although you can use Pascal's triangle from tables for most of the expansions you are likely to meet at this stage, there are times when you will have to use the formula for the binomial coefficients. If the value of n is large, the table may not show the row that you need.

If you go on to further study, you will learn to expand expressions involving a value of n that is not a positive whole number, such as –3 or ½. For values of n like these, you cannot use Pascal's triangle to find the binomial coefficients, so you have to use the formula. The work you are doing now is good preparation for this.

Example 6
In the expansion of $(3 - 2x)^8$, find the coefficient of

(i) x^2 (ii) x^5 (iii) x^7

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Solution

(i)   The term in x^2 is $_8C_2 \times 3^6 \times (-2x)^2 = 28 \times 729 \times 4x^2$

$$= 81648x^2$$

The coefficient of x^2 is 81648.

(ii)   The term in x^5 is $_8C_5 \times 3^3 \times (-2x)^5 = 56 \times 27 \times -32x^5$

$$= -48384x^5$$

The coefficient of x^5 is –48384.

(iii)   The term in x^7 is $_8C_7 \times 3^1 \times (-2x)^7 = 8 \times 3 \times -128x^7$

$$= -3072x^7$$

The coefficient of x^7 is –3072.