AQA Level 2 Further Mathematics Coordinate geometry

Section 2: Circles

Notes and Examples

These notes and examples contain subsections on

- The equation of a circle
- Finding the equation of a circle
- The intersection of a line and a circle
- Circle geometry

The equation of a circle

You should know about the following results, which you need to learn:

The general equation of a circle, centre the origin and radius r is

$$x^2 + y^2 = r^2$$

The general equation of a circle, centre (a, b) and radius r is

$$(x-a)^2 + (y-b)^2 = r^2$$

Make sure you understand why these equations describe circles.

Notice that the circle $(x-a)^2 + (y-b)^2 = r^2$ is just a translation of the circle $x^2 + y^2 = r^2$. The centre of the circle has moved from the point (0, 0) to the point (a, b). So the circle $x^2 + y^2 = r^2$ has been translated through the vector $\begin{pmatrix} a \\ b \end{pmatrix}$.

Example 1

For each of the following circles find (i) the coordinates of the centre and (ii) the radius.

(a) $x^2 + y^2 = 49$

(b) $(x + 2)^2 + (y - 6)^2 = 9$

Describe the transformation that maps the circle $(x + 2)^2 + (y - 6)^2 = 9$ to the circle $x^2 + y^2 = 9$

Solution

(a) $x^2 + y^2 = 49$ can be written as $x^2 + y^2 = 7^2$.

(i) The coordinates of the centre are (0, 0)

(ii) The radius is 7.

This is a particular case of the general form $x^2 + y^2 = r^2$ which has centre (0, 0) and radius r.

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(b) $(x + 2)^2 + (y - 6)^2 = 9$ can be written as $(x - (-2))^2 + (y - 6)^2 = 3^2$.

(i) The coordinates of the centre are (–2, 6)

(ii) The radius is 3.

A translation of $\begin{pmatrix} 2 \\ -6 \end{pmatrix}$.

This is a particular case of the general form $(x - a)^2 + (y - b)^2 = r^2$ which has centre (a, b) and radius r.

The centre of the circle is moved from (-2, 6) to (0, 0).

Finding the equation of a circle

You can write down the equation of a circle directly if you are given the coordinates of its centre and its radius.

Example 2

Find the equation of each of the following.

(a) a circle, centre (0, 0) and radius 4.

(b) a circle, centre (3, –4) and radius 6.

Solution

(a) The equation of a circle centre the origin is $x^2 + y^2 = r^2$

$r = 4$ so the equation is $x^2 + y^2 = 4^2$

i.e. $x^2 + y^2 = 16$

(b) The equation of a circle centre (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$

$a = 3$, $b = -4$ and $r = 6$ so the equation is $(x - 3)^2 + (y - (-4))^2 = 6^2$

i.e. $(x-3)^2 + (y+4)^2 = 36$

The intersection of a line and a circle

The next example looks at the intersection of a line and a circle.

Example 3

Find the coordinates of the point(s) where the circle $(x+2)^2 + (y-1)^2 = 9$ meets

(i) the line $y = 5$

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(ii) the line $x = 1$
(iii) the line $y = 2 - x$

Solution

(i) Substituting $y = 5$ into the equation of the circle:

$$(x+2)^2 + (5-1)^2 = 9$$
$$(x+2)^2 + 16 = 9$$
$$(x+2)^2 = -7$$

The expression $(x + 2)^2$ cannot be negative

There are no solutions. The line does not meet the circle.

(ii) Substituting $x = 1$ into the equation of the circle:

$$(1+2)^2 + (y-1)^2 = 9$$
$$9 + (y-1)^2 = 9$$
$$(y-1)^2 = 0$$
$$y = 1$$

The point is on the line $x = 1$, so its x-coordinate must be 1.

The line touches the circle at (1, 1).

(iii) Substituting $y = 2 - x$ into the equation of the circle:

$$(x+2)^2 + (2-x-1)^2 = 9$$
$$(x+2)^2 + (1-x)^2 = 9$$
$$x^2 + 4x + 4 + 1 - 2x + x^2 = 9$$
$$2x^2 + 2x - 4 = 0$$
$$x^2 + x - 2 = 0$$
$$(x-1)(x+2) = 0$$
$$x = 1 \text{ or } x = -2$$

Substitute the x values into the equation of the line to find the y-coordinates.

When $x = 1$, $y = 2 - 1 = 1$
When $x = -2$, $y = 2 - (-2) = 4$

The line crosses the circle at (1, 1) and (–2, 4).

Circle geometry

The four facts about circles given below are important. They often help to solve problems involving circles.

1. The angle in a semicircle is a right angle. In the diagram below AB is a diameter (with O the centre of the circle) and angles ADB and ACB are right angles (being subtended from the diameter to points on the circumference).

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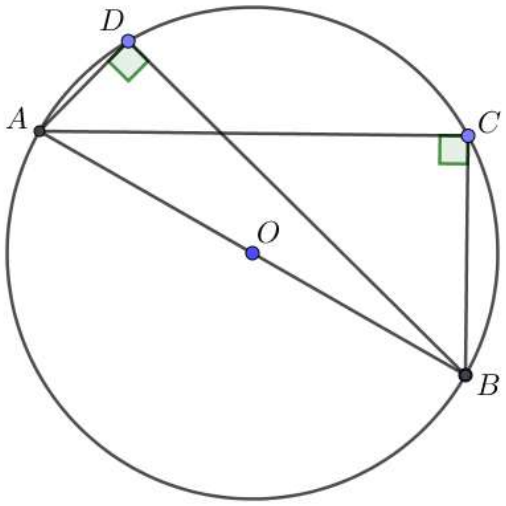
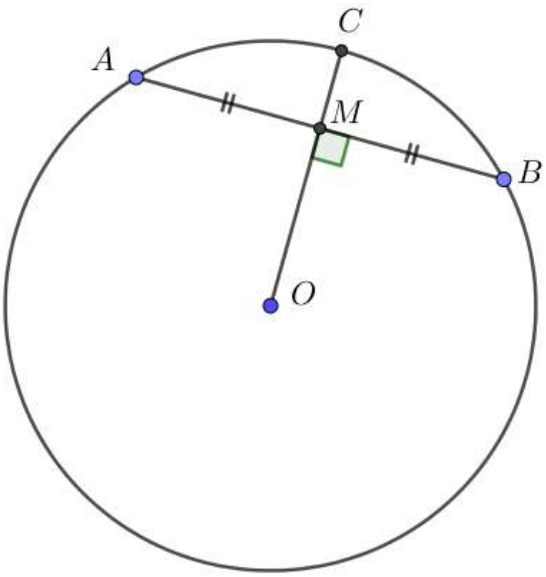
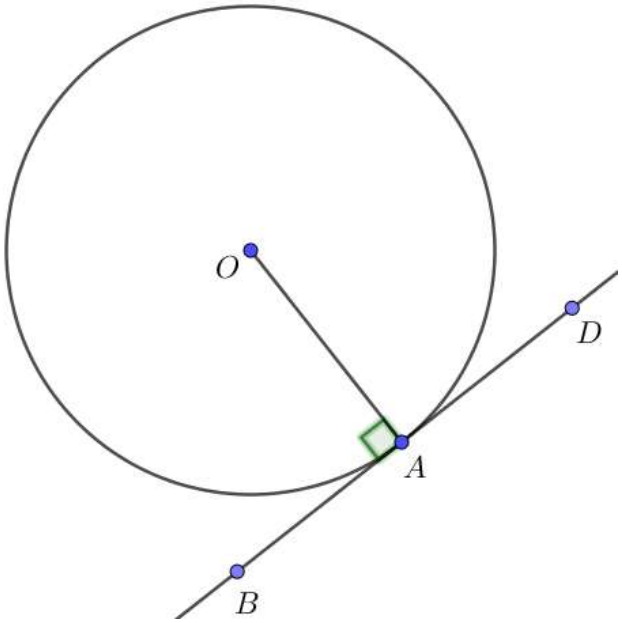


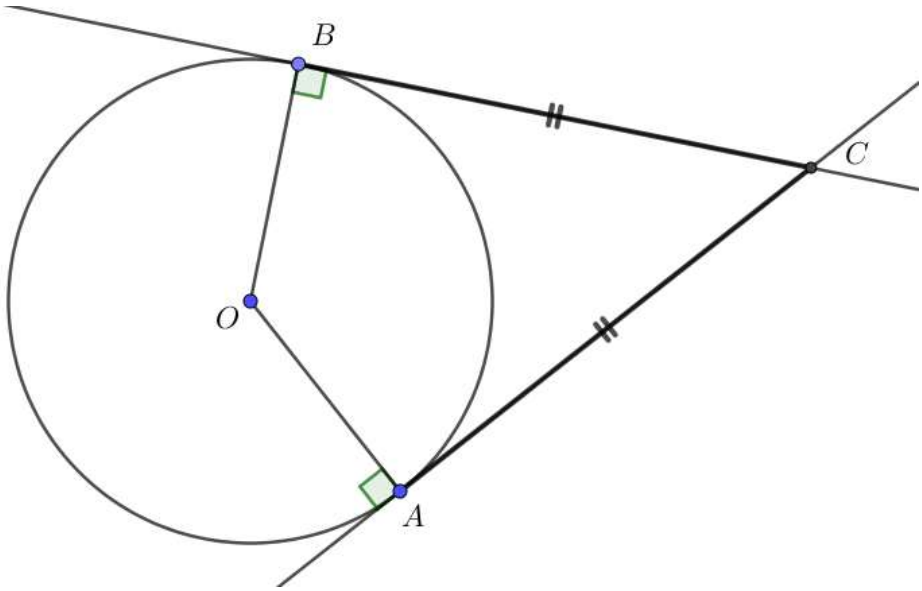
2. The perpendicular from the centre of a circle to a chord bisects the chord. In the diagram below, where O is the centre of the circle, the radius OC is perpendicular to the chord AB and it meets AB at the midpoint M of AB.



3. The tangent to a circle is perpendicular to the radius at that point. In the diagram below the line through B and D is tangent to the circle at A. The radius OA and this tangent are perpendicular.



4. Two tangents from a point to a circle are equal in length. In the diagram below the line passing through B and C is a tangent to the circle and the line passing through A and C is a tangent to the circle. The lengths BC and AC are equal.



Keep these properties in mind when dealing with problems involving circles.

Example 4
A circle is centred at the origin and goes through the point A = (2, -1).
Find the equation of the tangent to the circle at A

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Solution

The tangent is perpendicular to the line between the origin and A

Gradient of OA= $\frac{-1-0}{2-0} = -\frac{1}{2}$

Gradient of tangent = 2

The tangent goes through (2, -1) and has gradient 2

To find the gradient of a perpendicular line we find the negative reciprocal

$y - -1 = 2(x - 2)$

$y + 1 = 2x - 4$

$y = 3x - 5$

Example 5

D, E and F are points on a circle, DE is a diameter of the circle, D is on the y axis, E = (6,0) and F = (1, 5).

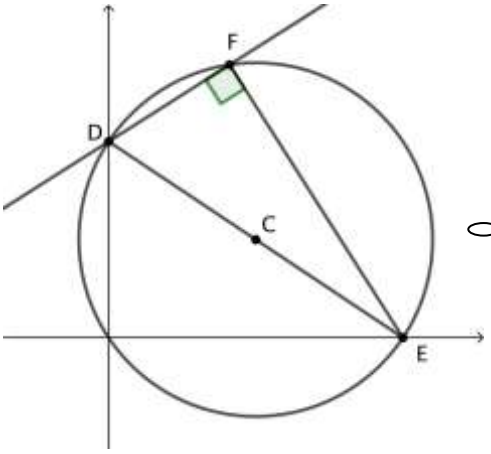
(i) Find the coordinates of point D.

(ii) Find the coordinates of the centre of the circle

(iii) Find the equation of the circle

Solution

(i) As DE is a diameter, angle DFE must be a right angle.



Drawing a diagram allows you to see where the right angles are and which lines are perpendicular

Gradient of EF = $\frac{0-5}{6-1} = \frac{-5}{5} = -1$

Gradient of DF = 1

Equation of line through DF :

$y - 5 = 1(x - 1)$

$y - 5 = x - 1$

$y = x + 4$

As D is on the y axis the coordinates are where this line intercepts the y axis.

This will be at (0, 4)

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(ii) The centre of the circle is the midpoint of the diameter DE.

$$\left(\frac{6+0}{2}, \frac{0+4}{2}\right) = (3,2)$$

(iii) Thr radius is the distance from the centre to any point on the edge.
For example (3, 2) to (0, 4)

$$\sqrt{(3-0)^2 + (2-4)^2}$$

$$\sqrt{(3)^2 + (-2)^2} = \sqrt{9+4} = \sqrt{13}$$

Equation of the circle: $(x-3)^2 + (y-2)^2 = 13$

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