Section 2: Graphs of functions

Notes and Examples

These notes contain subsections on:

- Gradients
- The equation of a straight line
- The intersection of two lines
- Sketching the graphs of functions

Gradients

To find the gradient of a straight line between two points (x_1, y_1) and (x_2, y_2), use the formula

$$\text{gradient } = \frac{y_2 - y_1}{x_2 - x_1}.$$

Example 3
P is the point (-3, 7). Q is the point (5, 1).
Calculate the gradient of PQ

Solution
Choose P as (x_1, y_1) and Q as (x_2, y_2).

or vice versa: it will still give the same answer (***WHY***?)

$$\text{Gradient of PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-7}{5-(-3)} = \frac{-6}{8} = -\frac{3}{4}$$

Notes:
(1) Draw a sketch and check that your answer is sensible (e.g. has negative gradient).
(2) Check that you get the same result when you choose Q as (x_1, y_1) and P as (x_2, y_2).

The equation of a straight line

The equation of a straight line is often written in the form $y = mx + c$, where m is the gradient and c is the intercept with the y-axis.

Example 4
Find (i) the gradient and (ii) the y-intercept of the following straight-line equations.
(a) $5y = 7x - 3$ (b) $3x + 8y - 7 = 0$

Solution

(a) Rearrange the equation into the form $y = mx + c$.

$5y = 7x - 3$ becomes $y = \frac{7}{5}x - \frac{3}{5}$

so $m = \frac{7}{5}$ and $c = -\frac{3}{5}$

Note the minus sign

(i) The gradient is $\frac{7}{5}$

(ii) The y-intercept is $-\frac{3}{5}$.

(b) Rearrange the equation into the form $y = mx + c$.

$3x + 8y - 7 = 0$ becomes $8y = -3x + 7$

giving $y = -\frac{3}{8}x + \frac{7}{8}$

so $m = -\frac{3}{8}$ and $c = \frac{7}{8}$

Note the minus sign

(i) The gradient is $-\frac{3}{8}$

(ii) The y-intercept is $\frac{7}{8}$.

Sometimes you may need to sketch the graph of a line. A sketch is a simple diagram showing the line in relation to the origin. It should also show the coordinates of the points where it cuts one or both axes.

Example 5

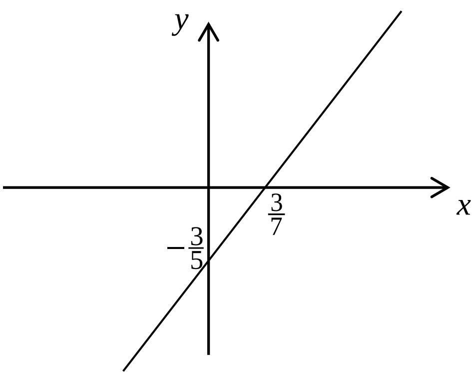
Sketch the lines (a) $5y = 7x - 3$ (b) $3x + 8y - 7 = 0$

Solution

(a) $5y = 7x - 3$

When $x = 0$, $y = -\frac{3}{5}$

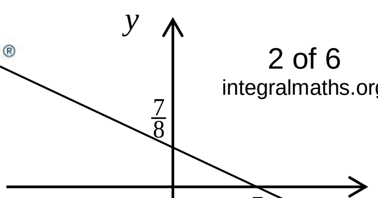
When $y = 0$, $x = \frac{3}{7}$



(b) $3x + 8y - 7 = 0$

When $x = 0$, $y = \frac{7}{8}$

When $y = 0$, $x = \frac{7}{3}$



Sometimes you may need to find the equation of a line given certain information about it. If you are given the gradient and intercept, this is easy: you can simply use the form $y = mx + c$. However, more often you will be given the information in a different form, such as the gradient of the line and the coordinates of one point on the line (as in Example 6) or just the coordinates of two points on the line (as in Example 7).

In such cases you can use the alternative form of the equation of a straight line. For a line with gradient m passing through the point (x_1, y_1), the equation of the line is given by

$$y - y_1 = m(x - x_1).$$

Example 6
Find the equation of the line with gradient 2 and passing through (3, -1).

Solution
The equation of the line is $\quad y - y_1 = m(x - x_1)$

$m = 2$ and (x_1, y_1) is (3, -1)

$$\Rightarrow y - (-1) = 2(x - 3)$$
$$\Rightarrow y + 1 = 2x - 6$$
$$\Rightarrow y = 2x - 7$$

You should check that the point (3, -1) satisfies your line. If it doesn't, you must have made a mistake!

In the next example, you are given the coordinates of two points on the line.

Example 7
P is the point (3, 8). Q is the point (-1, 5).
Find the equation of PQ.

One method is to find the gradient and then use this value and one of the points in $y - y_1 = m(x - x_1)$

Solution

Choose P as (x_1, y_1) and Q as (x_2, y_2).

$$\text{Gradient of PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-8}{-1-3} = \frac{-3}{-4} = \frac{3}{4}$$

Now use $y - y_1 = m(x - x_1)$

$$\Rightarrow y - 8 = \tfrac{3}{4}(x-3)$$
$$\Rightarrow 4(y-8) = 3(x-3)$$
$$\Rightarrow 4y - 32 = 3x - 9$$
$$\Rightarrow 4y = 3x + 23$$

You should check that P and Q satisfy your line.

An alternative approach to the above examples is to put the formula for m into the straight line equation to obtain

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

and then make the substitutions. This is equivalent to the first method, but does not involve calculating m separately first.

The intersection of two lines

The point of intersection of two lines is found by solving the equations of the lines simultaneously. This can be done in a variety of ways. When both equations are given in the form $y = \ldots$ then equating the right hand sides is a good approach (see below). If both equations are not in this form, you can re-arrange them into this form first, then apply the same method. Alternatively, you can use the elimination method if the equations are in an appropriate form.

Example 8

Find the point of intersection of the lines $y = 3x - 2$ and $y = 5x - 8$.

<u>Solution</u>

$$3x - 2 = 5x - 8$$
$$\Rightarrow -2 = 2x - 8$$
$$\Rightarrow 6 = 2x$$
$$\Rightarrow x = 3$$

Substitute into one of the equations to find y

Substituting $x = 3$ into $y = 3x - 2$ gives $y = 3 \times 3 - 2 = 7$

The point of intersection is (3, 7)

Check that (3, 7) satisfies the second equation.

Sketching the graphs of functions

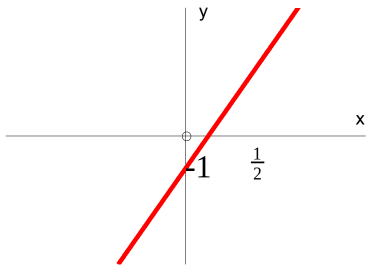
You can sketch the graph of a linear or quadratic function by thinking about where the graph cuts the coordinate axes.

- A linear function is of the form $f(x) = ax + b$. The graph of $y = f(x)$ is a straight line. You can find where it cuts the y-axis by substituting $x = 0$, and you can find where it cuts the x-axis by substituting $y = 0$, as in Example 5.

- A quadratic function is of the form $f(x) = ax^2 + bx + c$. The graph of $y = f(x)$ is a curve called a parabola. You can find where the graph cuts the y-axis by substituting $x = 0$. You can find where it cuts the x-axis by solving the equation $ax^2 + bx + c = 0$. If the equation has no solutions, then the graph does not cut the x-axis.

Example 9
Sketch the graph of $y = f(x)$ for each of the following functions:
(i) $f(x) = 2x - 1$
(ii) $f(x) = x^2 - x - 2$
(iii) $f(x) = x^2 + 1$

Solution
(i) This is a straight line graph.
$f(0) = -1$ so the graph crosses the y-axis at (0, -1)
When $y = 0$, $2x - 1 = 0$ so $x = \frac{1}{2}$, so the graph crosses the x-axis at $\left(\frac{1}{2}, 0\right)$.

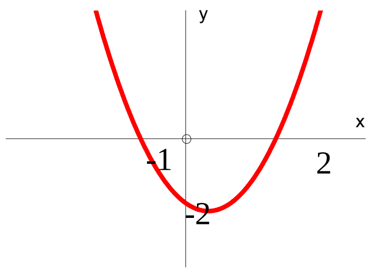


(ii) $f(0) = -2$, so the graph crosses the y-axis at (0, -2)
When $y = 0$, $x^2 - x - 2 = 0$

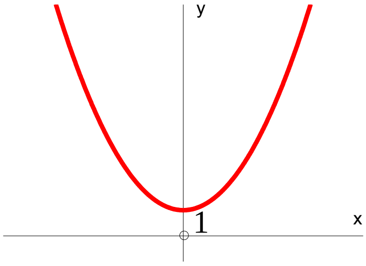
$$(x-2)(x+1) = 0$$

$$x = 2 \text{ or } -1$$

so the graph crosses the x-axis at (2, 0) and (-1, 0)



(iii) f(0) = 1, so the graph crosses the y-axis at (0, 1)
When $y = 0$, $x^2 + 1 = 0$ has no solutions, so the graph does not cut the x-axis.



Functions are sometimes defined 'piecewise' – this means that the domain is split into separate parts and the function is defined differently for each part. Here is an example.

Example 10
A function f(x) is defined as

$$\begin{aligned} f(x) &= 1 & x &< 0 \\ &= x + 1 & 0 \le x &< 3 \\ &= 7 - x & x &\ge 3 \end{aligned}$$

Sketch the graph of $y = f(x)$

Solution