AQA Level 2 Further Mathematics Geometry II

Section 1: Triangles, sine, cosine rule

Notes and Examples

These notes contain sub-sections on:
- **Trigonometric functions for angles of any size**
- **Graphs of trigonometric functions**
- **Common values of sin *x*, cos *x* and tan *x***
- **The sine rule**
- **The cosine rule**
- **Choosing which rule to use**
- **The area of a triangle**

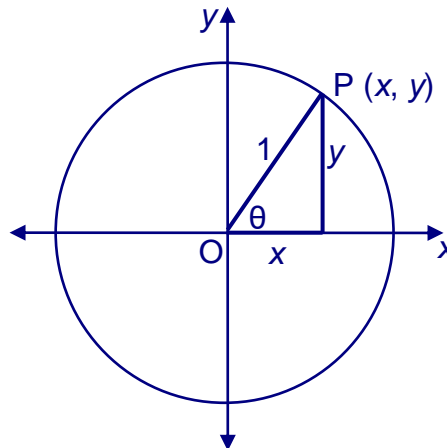## Trigonometric functions for angles of any size

The definitions for sine, cosine and tangent can be extended to angles of any size using a diagram like the one below.

This gives the definitions:

$$\sin\theta = \frac{y}{1} = y$$

$$\cos\theta = \frac{x}{1} = x$$

$$\tan\theta = \frac{y}{x}$$

- In the first quadrant, θ lies between 0° and 90°. The values of x and y are both positive, so the values of sin θ, cos θ and tan θ are all positive.
- In the second quadrant, θ lies between 90° and 180°. The value of x is negative and the value of y is positive, so sin θ is positive but cos θ and tan θ are both negative.
- In the third quadrant, θ lies between 180° and 270. The values of x and y are both negative, so tan θ is positive but sin θ and cos θ are both negative.
- In the fourth quadrant, θ lies between 270° and 360°. The values of x is positive and the value of y is negtive, so cos θ is positive but sin θ and tan θ are both negative.
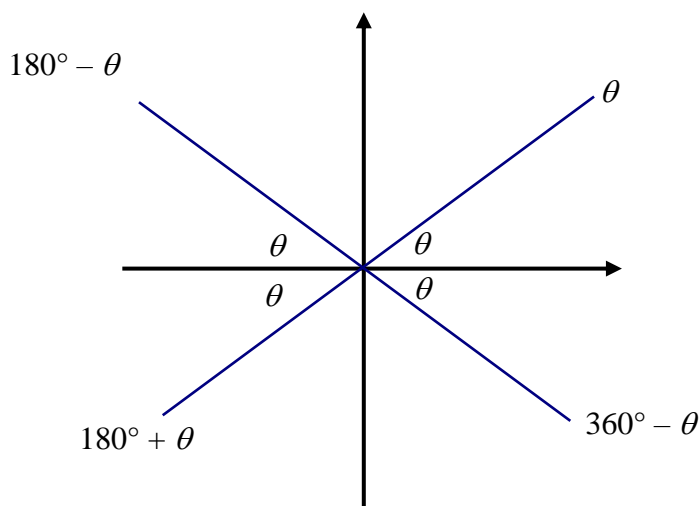
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This can be summarised in the diagram below.

Sine positive	All positive
Tangent positive	**C**osine positive

or, more simply

S	A
T	C

This is often known as the "CAST" diagram: some people remember it using the word **CAST** (make sure you start the word in the right place!); others use a phrase such as "**A**ll **S**chool **T**eachers **C**riticise" (which, while rather unfair to the teaching profession, does have the advantage of starting in the first quadrant).

The symmetry of the diagram allows you to find relate trig rations of angles in the second, third and fourth quadrant to trig ratios angles in the first quadrant.

To find a trig ratio of an angle in the second, third or fourth quadrant in terms of the trig ratio of an acute angle, you need to find the equivalent acute angle using symmetry, and then use the CAST diagram to find whether the trig ratio is positive or negative.
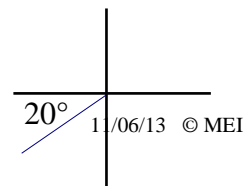
It is usually helpful to draw a sketch diagram, as shown in the example below.

Example 1
Find, in terms of a trigonometric ratio of an acute angle:
(i) $\sin 200°$
(ii) $\cos 135°$
(iii) $\tan 255°$

$20°$

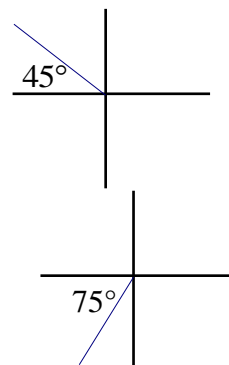**Solution**
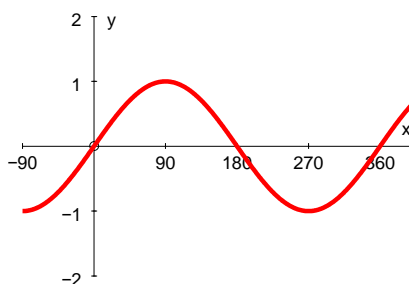
(i) 200° is in the third quadrant, so has a negative sine.

$$\sin 200° = -\sin(200° - 180°)$$
$$= -\sin 20°$$

(ii) 135° is in the second quadrant, so has a negative cosine.

$$\cos 135° = -\cos(180° - 135°) = -\cos 45°$$

(iii) 255° is in the third quadrant, so has a positive tangent.

$$\tan 255° = \tan(180° + 75°)$$
$$= \tan 75°$$

Graphs of trigonometric functions

Now that you can define sin x, cos x and tan x for any value of x between 0° and 360° you can draw the graphs of these functions:
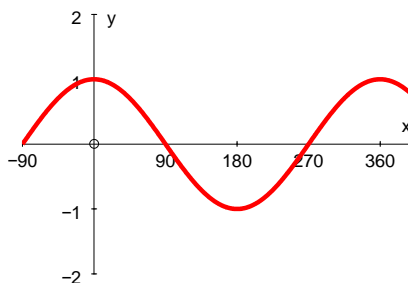
Graph of $y = \sin x$

You only need to know the shape of the graph for $0° \le x \le 360°$. However, the graph repeats every 360°.

Notice that $y = \sin x$ lies between 1 and -1.

Graph of $y = \cos x$

You only need to know the shape of the graph for $0° \leq x \leq 360°$. However, the graph repeats every 360°.

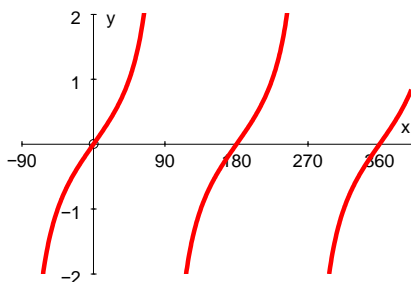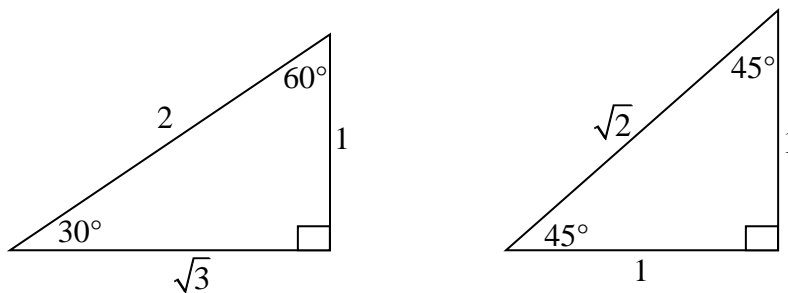Notice that $y = \cos x$ lies between 1 and -1.

Graph of $y = \tan x$

You only need to know the shape of the graph for $0° \leq x \leq 360°$. However, the graph repeats every 180°.

There are asymptotes at $x = 90°$ and $x = 270°$, where the value of $\tan x$ is undefined (since $\cos x = 0$ for these values of x)

Common values of sin *x*, cos *x* and tan *x*

In the Geometry section you saw how two special triangles can be used to find the values of sin *x*, cos *x* and tan *x* for *x* = 30°, 45° and 60°.

You should also be able to remember the values of sin *x*, cos *x* and tan *x* for *x* = 0° and 90°.

All these values are summarised in the table below.

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x	0°	30°	45°	60°	90°
cos x	1	$\dfrac{\sqrt{3}}{2}$	$\dfrac{1}{\sqrt{2}}$	$\dfrac{1}{2}$	0
sin x	0	$\dfrac{1}{2}$	$\dfrac{1}{\sqrt{2}}$	$\dfrac{\sqrt{3}}{2}$	1
tan x	0	$\dfrac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

If you find it hard to remember the values for 30°, 45° and 60° at first, you can

In the example below you need to substitute exact values into an expression.

Example 2
Show that $\sin^2 30° + \cos^2 30° = 1$

Solution
$\sin 30° = \dfrac{1}{2}$ and $\cos 30° = \dfrac{\sqrt{3}}{2}$

So substituting these values into $\sin^2 30° + \cos^2 30°$:

$$\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1 \qquad \text{as required.}$$

You can find the values of other angles in the other three quadrants, that are related to the angles 30°, 45° and 60°. This is shown in Example 3.
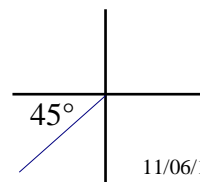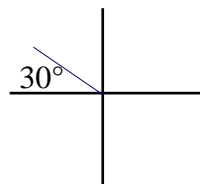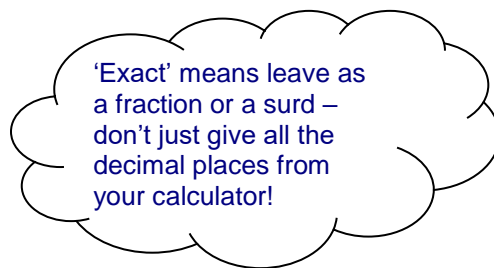
Example 3
Write down the exact values of
(i)   tan 150°
(ii) sin 225°
(iii) cos 300°

> 'Exact' means leave as a fraction or a surd – don't just give all the decimal places from your calculator!

Solution
(i) 150° is in the second quadrant, so tan is negative
$\tan 150° = -\tan(180° - 150°)$

$\qquad = -\tan 30°$

$\qquad = -\dfrac{1}{\sqrt{3}}$

30°

45°

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(ii) $225°$ is in the third quadrant, so sin is negative

$$\sin 225° = -\sin(225° - 180°)$$

$$= -\sin 45°$$
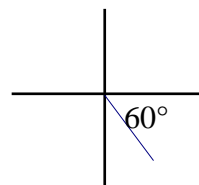
$$= -\frac{1}{\sqrt{2}}$$

(iii) $300°$ is in the fourth quadrant, so cos is positive

$$\cos 300° = \cos(360° - 300°)$$

$$= \sin 60°$$

$$= \frac{1}{2}$$

The sine rule

If a triangle does not have a right-angle, you can find missing sides and angles using the **sine rule** or the **cosine rule**.

The sine rule:

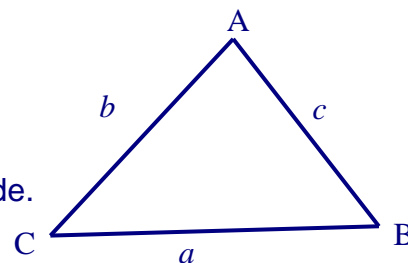$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

This form is easier to use when finding an unknown side.

The sine rule can also be written as:

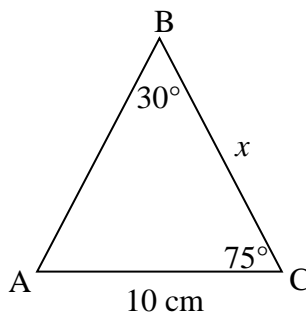$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

This form is easier to use when finding an unknown angle.
Example 4 shows the use of the sine rule to find an unknown side.

Example 4
Find the side BC in the triangle ABC.

Solution

By the sine rule:

$$\frac{x}{\sin A} = \frac{b}{\sin B}$$

So:

$$\frac{x}{\sin 75°} = \frac{10}{\sin 30°}$$

$$\Rightarrow x = \frac{10 \sin 75°}{\sin 30°}$$

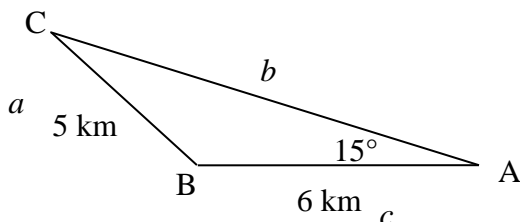so $x = 19.3$ cm (to 3 sig.fig.)

Example 5 shows the use of the sine rule to find an unknown angle.

Notice that when finding an angle, sometimes there is more than one possible solution.

Example 5
Find $\angle ACB$ in the diagram below.

Solution

By the sine rule: $\dfrac{\sin A}{a} = \dfrac{\sin C}{c}$

So: $\dfrac{\sin 15°}{5} = \dfrac{\sin C}{6}$

$\Rightarrow$ $\sin C = \dfrac{6 \sin 15°}{5}$

> Don't round here! Store the number in your calculator.

$\sin C = 0.310....$
$C = 18.1°$ to 1 d.p.

Check whether $180° - C$ is also a solution:
            $180° - 18.1° = 161.9°$ to 1 d.p.

> Angles A and C still add up to less than 180°

This also works so $\angle ACB$ is 18.1° to 1 d.p. or 161.9° to 1 d.p

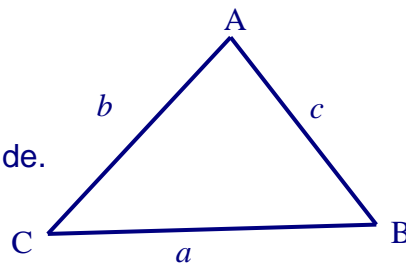## The cosine rule

The cosine rule:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

This form is easier to use when finding an unknown side.

The cosine rule can also be written as:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

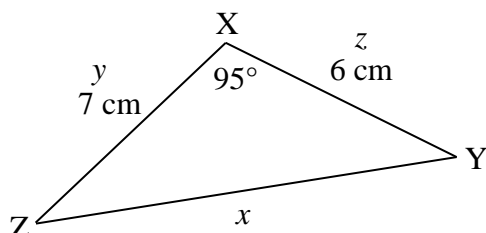This form is easier to use when finding an unknown angle.

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Example 6 shows the use of the cosine rule to find an unknown side.

Example 6
Find the side YZ in the triangle XYZ.

Solution
The cosine rule for this triangle is:

$$x^2 = y^2 + z^2 - 2yz\cos X$$

So:

$$x^2 = 7^2 + 6^2 - 2 \times 7 \times 6 \cos 95°$$

$$x^2 = 92.32...$$
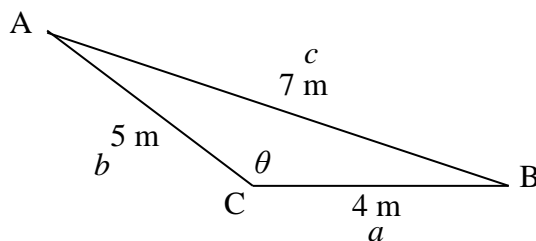
$$x = 9.61 \text{ cm to 3 sig. fig.}$$

Example 7 shows the use of the cosine rule to find an unknown angle.

Example 7
Find the angle θ in the triangle ABC.

Solution

The cosine rule for this triangle is:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos C = \frac{4^2 + 5^2 - 7^2}{2 \times 4 \times 5}$$

$$\cos C = -0.2$$

$$C = 101.5° \text{ to 1 d.p.}$$

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Choosing which rule to use

Use the sine rule when:

- you know 2 sides and 1 angle (not between the two sides) and want a 2nd angle (3rd angle is now obvious!)
- you know 2 angles and 1 side and want a 2nd side

Use the cosine rule when:

- you know 3 sides and want any angle
- you know 2 sides and the angle between them and want the 3rd side

Example 8 shows how to decide whether to use the sine or the cosine rule.
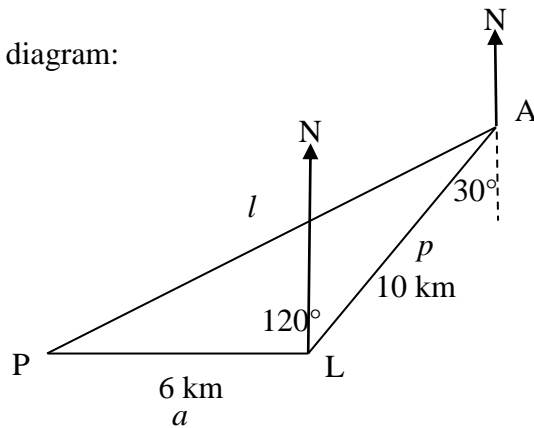
Example 8
A ship sails from a port, P, 6 km due East to a lighthouse, L, 6 km away.
The ship then sails 10 km on a bearing of 030° to an island, A.
Find: (i)   The distance AP
        (ii)   The bearing of P from A

Solution
First draw a diagram:

(i)   You know 2 sides and the angle between them so you need the cosine rule.
     The cosine rule for this triangle is:
     $$l^2 = a^2 + p^2 - 2ap\cos l$$
     $$l^2 = 6^2 + 10^2 - 2\times6\times10\cos120°$$
     $$l^2 = 196$$
     $$l = 14 \text{ km}$$

     So the distance AP is 14 km.

(ii) You can now use either the cosine rule or the sine rule to find the angle PAL.

     The sine rule for this triangle is: $\dfrac{\sin A}{a} = \dfrac{\sin L}{l}$

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So:
$$\frac{\sin A}{6} = \frac{\sin 120°}{14}$$

$$\therefore \sin A = \frac{6 \sin 120°}{14}$$

$$\therefore \sin A = 0.371....$$

$$A = 21.8°$$

So the bearing is $180° + 30° + 21.8° = 231.8°$ to 1 d.p.

> Check whether $180° - 21.8° = 158.2°$ is also a solution.
> It isn't because the angles in the triangle would total more than $180°$.

The area of a triangle

To find the area of any triangle you can use the rule:

$$\text{Area of triangle ABC} = \frac{1}{2} ab \sin C$$

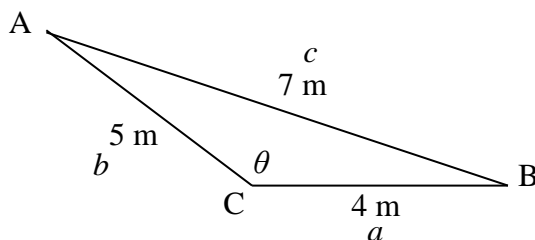> So you need two sides and the angle between them.

Example 9 shows how to use this formula.

Example 9
Find the area of triangle ABC from Example 7.

Solution

In Example 4, angle C was found to be $101.5°$ to 1 d.p.

Using the formula Area $= \frac{1}{2} ab \sin C$ gives:

Area of triangle ABC $= \frac{1}{2} \times 4 \times 5 \times \sin 101.5...° = 9.80 \, \text{m}^2$