AQA Level 2 Further Mathematics Algebra III

Section 1: Functions

Notes and Examples

These notes contain subsections on:

- The language of functions
- Composition of functions
- Inverse functions

The language of functions

"Start with any positive number. Multiply by 2, and then add 1".

The rule above is an example of a function. A function is a rule which can be applied to any object or number in a particular set, and outputs a new object or number.

When you are working with numbers, a function can usually be expressed algebraically. The function above can be written like this:

$$f(x) = 2x + 1 \qquad x > 0$$

The set of possible inputs for the function is called the domain of the function. In this case, the domain is the set of all positive numbers, so this can be written as $x > 0$.

The set of possible outputs for the function is called the range of the function. In this case, because the domain is $x > 0$, the output must be greater 1, so the range is $f(x) > 1$. If we had chosen a different domain, the range would have been different. For example, we could have chosen "all real numbers" for the domain, in which case the range would also be "all real numbers".

Example 1

The function f(x) is defined as $f(x) = (x - 1)^2 + 3 \qquad x > 0$

(i) Find the value of f(5).

(ii) What is the range of f(x)?

(iii) Find an expression for $f(2x + 1)$. Give your answer in its simplest form.

Solution

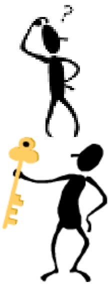
(i) $f(5) = (5 - 1)^2 + 3 = 4^2 + 3 = 16 + 3 = 19$.

(ii) The smallest possible value of f(x) is 3 (when $x = 1$). So the range of f(x) is given by $f(x) > 3$.

The expression in the bracket cannot be negative – its smallest possible value is 0

(iii) $$\begin{aligned} f(2x+1) &= (2x+1-1)^2 + 3 \\ &= (2x)^2 + 3 \\ &= 4x^2 + 3 \end{aligned}$$

Replace x by $2x + 1$ in the definition of the function



Example 2

The function f(x) is defined as $f(x) = \dfrac{x}{x+1}$.

(i) Find f(4).

(ii) f(x) has domain all x except $x = a$. What is the value of a?

Solution

(i) $f(4) = \dfrac{4}{4+1} = \dfrac{4}{5} = 0.8$

(ii) $a = -1$

When $x = -1$, the denominator is zero so the function isn't defined. So $x = -1$ has to be excluded from the domain.

Composition of functions

Suppose you have two functions f(x) and g(x). If the range of g(x) is contained in the domain of f(x), you apply g then f to any value of x in the domain of g. This is known as composing functions.

When the range of g is contained in the domain of f the function f∘g is defined as follows

$$f \circ g(x) = f(g(x)).$$

Suppose the $f(x) = 2x + 3$ and $g(x) = x - 2$.

f is the function 'multiply by 2 and then add 3'. g is the function 'subtract 2'.

To apply function f∘g you apply g and then f. This means that you 'subtract 2, then multiply by 2 and then add 3. As an expression this is

$2(x – 2) + 3$.

You might write this down as follows

$f \circ g(x) = f(g(x)) = f(x - 2) = 2(x - 2) + 3$.

By multiplying out the brackets, this can be simplified to $f \circ g(x) = 2x - 1$.

Notice that the function g∘f is different. This is 'f then g'. This means you 'multiply by 2, add 3, then subtract 2. As an expression this is

$2x + 3 – 2$

You might write this down as follows

$$g \circ f(x) = g(f(x)) = g(2x + 3) = 2x + 3 - 2 = 2x + 1.$$

So $f \circ g(x) = 2x - 1$ and $g \circ f(x) = 2x + 1$ they are different functions. For example $f \circ g(1) = 1$ and $g \circ f(1) = 3$

Here are some more examples.

Example 3

Let $f(x) = 2x$, $g(x) = x^2$ and $h(x) = \dfrac{x}{x+1}$. Find formulae for the following functions

(i) $f \circ g$
(ii) $g \circ f$
(iii) $f \circ h$
(iv) $g \circ h$

Solution

(i) $f \circ g$ means "apply g and then apply f". So $f \circ g(x) = f(g(x)) = f(x^2) = 2x^2$

(ii) $g \circ f$ means "apply f and then apply g". So $g \circ f(x) = g(f(x)) = g(2x) = (2x)^2 = 4x^2$

(iii) $f \circ h$ means "apply h and then apply f". So $f \circ h(x) = f(h(x)) = f\left(\dfrac{x}{x+1}\right) = \dfrac{2x}{x+1}$

(iv) $g \circ h$ means "apply h and then apply g". So $g \circ h(x) = g(h(x)) = g\left(\dfrac{x}{x+1}\right) = \left(\dfrac{x}{x+1}\right)^2$

Note that in examples (iii) and (iv) the value $x = -1$ has to be excluded from the domain of the function. This is because h is the first function being applied in these two examples and h itself is not defined at $x = -1$.

Sometimes you might need to think even more carefully about the domain of a composed function, as in the next example.

Example 4

Let $f(x) = 2x$, $h(x) = \dfrac{x}{x+1}$. The domain of $h \circ f$ is all values of x other than $x = a$. What is the value of a?

Solution

h is not defined when $x = -1$. Since you are applying f first in this examples the composite function will not be defined for values a such that $f(a) = -1$.

This means that $2a = -1$ and $a = -\dfrac{1}{2}$.

Here is another way to think about this

h∘f means "apply f and then apply h". So $h \circ f(x) = h(f(x)) = h(2x) = \frac{2x}{2x+1}$. Looking at the denominator of this expression it can be seen that the expression is not defined when $x = -\frac{1}{2}$.

So $a = -\frac{1}{2}$.

Inverse functions

Think about the following two functions f(x) and g(x).

$f(x) = 2x - 1, \qquad g(x) = \frac{x+1}{2}$

Think about the function g∘f.

This means 'apply f, then apply g'.

This will be 'multiply by 2, then subtract 1, then add 1, then divide by 2'.

Think carefully about this, convince yourself that if you did this to any number then the value would be unchanged.

Now think about the function f∘g.

This means 'apply g, then apply f'.

This will be 'add 1, then divide by 2, then multiply by 2, then add 1'.

Again, think carefully about this, convince yourself that if you did this to any number then the value would be unchanged.

This means that both g∘f and f∘g leave values unchanged. When two functions f and g have this relationship to one another you say they are inverse functions to one another. f is the inverse of g and g is the inverse of f.

Sometimes you write f^{-1} for the function that is the inverse of f. So, in the example above $g = f^{-1}$ and $f = g^{-1}$

Given a function you can try to work out if it has an inverse function and what it would be. Here is a way to do this.

Suppose you are looking for an inverse function to $f(x) = 3x + 4$.

The inverse function needs to take the output of f, $3x + 4$ and 'turn it back into x'.

To do this you would need to 'subtract 4 and then divide by 3.

This means that the inverse function to f is $g(x) = \dfrac{x-4}{3}$

Another way to do this is as follows.

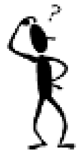
Let y be the output from f, so $y = 3x + 4$.

Now switch y and x around, so $x = 3y + 4$.

Now make y the subject of this equation,

$$x = 3y + 4 \Rightarrow x - 4 = 3y \Rightarrow y = \frac{x-4}{3}$$

As before, this means that the inverse function to f is $g(x) = \dfrac{x-4}{3}$



Example 5
Find the inverse function to each of the following functions
(i) $f(x) = 3x - 1$, where x can take any value
(ii) $f(x) = \dfrac{3}{x}$, where x can take any value other than zero.

Solution
(i) To undo f which is 'multiply by 3 and then subtract 1' you need to 'add 1 and then divide by 3'. This means that $f^{-1}(x) = \dfrac{x+1}{3}$.

Alternatively let $y = 3x - 1$, switch x and y and then make y the subject as follows.

$$x = 3y - 1 \Rightarrow 3y = x + 1 \Rightarrow \frac{x+1}{3}$$. Again $f^{-1}(x) = \dfrac{x+1}{3}$

(ii) To undo f which is 'invert (take the reciprocal) and then multiply by 3 you need to 'divide by 3 then invert. This means that $f^{-1}(x) = \dfrac{3}{x}$. Note that it's the same function as f, it is not defined when $x = 0$ either.

Alternatively let $y = \dfrac{3}{x}$, switch x and y and then make y the subject as follows.

$$x = \frac{3}{y} \Rightarrow y = \frac{3}{x}$$. Again $f^{-1}(x) = \dfrac{3}{x}$, where $x \neq 0$.

In the final example you'll see that sometimes particular care is needed with domains when thinking about inverses.

As a function on all the real number $f(x) = x^2$ has no inverse. This is because it is not one-to-one with this domain. For example $f(2)$ and $f(-2)$ are both 4. An inverse can't exist when f has this domain because there are two possible values that it should take when $f(x) = 4$.

Now think about $f(x) = x^2$ when the domain is restricted to real numbers which are greater than or equal to zero. The function is now one to one and an inverse exists which is $g(x) = \sqrt{x}$ ($g(x)$ is the positive square root of x).

Example 6

Find the inverse of the function $f(x) = (x-2)^2$ with domain $x \geq 2$.

Solution

Note that without the restriction on the domain this function would not have an inverse because f(1) = f(3) = 1, for example, and so the function is not one to one if its domain is all the real numbers.

To undo 'subtract 2 and then square' you need to 'square root and then add 2'. So the inverse function is $f^{-1}(x) = \sqrt{x} + 2$.

Alternatively let $y = (x-2)^2$, switch x and y and then make y the subject as follows.

$x = (y-2)^2 \Rightarrow y - 2 = \sqrt{x} \Rightarrow y = \sqrt{x} + 2$. Again $f^{-1}(x) = \sqrt{x} + 2$.