Section 6: Sequences and proof

Notes and Examples

These notes contain subsections on

- **Algebraic proof**
- **Sequences**
- **The limit of a sequence**

Algebraic proof

Proof is a very important aspect of mathematics. In this course you are expected to have an idea about what proof involves and to carry out simple algebraic proofs. In later topics you will also look at geometrical proofs.

The most important thing to realise is that checking lots of cases does not prove that the result is true. As a very simple example, think of three consecutive numbers and add them up. You should find that this sum is divisible by 3. Suppose you want to prove that the sum of three consecutive integers is always divisible by 3. You could test quite a lot of sets of numbers yourself, or you could program a computer to test a very large number of sets of numbers. The computer could keep checking numbers up to astronomically large numbers, but you would still not have checked every single number, and you never can, since there are an infinite number of sets of three consecutive integers! At this stage you could feel sure that the conjecture is in fact true, but to prove it you need to show that it is true for all possible sets of numbers.

Fortunately, this is very easy to do.

Example 1
Prove that the sum of any three consecutive integers is divisible by 3.

Solution
Let the first number be n.
Then the second number is $n + 1$, and the third number is $n + 2$.
The sum of the three numbers is $n + n + 1 + n + 2 = 3n + 3 = 3(n + 1)$
$3(n + 1)$ is divisible by 3 for all values of n.

When you write out a proof, you must make sure that you show every step clearly. Sometimes your steps are shown in algebra, and sometimes you need to use words to explain something.

Sometimes there can be more than one way to prove something. The next example gives two different proofs of the same result.

Example 2
n is a positive integer. Prove that $n^2 + n$ is always even.

Solution 1
If n is odd, n^2 is odd, so $n^2 + n$ is the sum of two odd numbers and is therefore even.
If n is even, n^2 is even, so $n^2 + n$ is the sum of two even numbers and is therefore even.

Solution 2
Factorising: $n^2 + n = n(n + 1)$
If n is even, $n + 1$ is odd.
If n is odd, $n + 1$ is even.
So $n(n + 1)$ is the product of an odd number and an even number, and so it is even for all integer values of n.

Sequences

A sequence is a set of numbers in a given order. These numbers may form an algebraic pattern.

Sequences which follow a pattern can be defined algebraically by using a formula for the nth term. The terms of the sequence can be found by substituting the numbers 1, 2, 3… for n.

Example 3
The nth term of a sequence is given by $n^2 - 3$
(i) Write down the first five terms of the sequence.
(ii) Find the 20th term of the sequence.

Solution
(i) Substituting $n = 1$, $n = 2$, … $n = 5$ into the expression $n^2 - 3$ gives the terms of the sequence:

1st term $= 1^2 - 3 = -2$
2nd term $= 2^2 - 3 = 1$
3rd term $= 3^2 - 3 = 6$
4th term $= 4^2 - 3 = 13$
5th term $= 5^2 - 3 = 22$

The first five terms of the sequence are $-2, 1, 6, 13, 22$.

(ii) Substituting $n = 20$
20th term $= 20^2 - 3 = 400 - 3 = 397$

Sometimes you may be given a sequence and be asked to find the formula for the nth term.

If the sequence is linear, the difference between the terms is the always the same. A linear sequence has nth term given by $an + b$. You will need to find the values of a

and b. It's easy to find the value of a, as it is equal to the difference between one term and the next. (Think about why this is true). Once you know a, you can find the value of b which gives the correct terms in the sequence.

Example 4
The first five terms of a linear sequence are 1, 4, 7, 10, 13.
Find an expression for the nth term.

Solution
The difference between terms is 3.
So the nth term is $3n + b$.
The first term is 1, so $3\times1+b=1$

$$3+b=1$$

$$b=-2$$

The nth term is $3n - 2$.

Finding the formula for the nth term of a quadratic sequence is a little more difficult. A quadratic sequence has nth term of the form $an^2 + bn + c$.

You can recognise a quadratic sequence by looking at the difference between one term and the next. The differences go up (or down) by the same number each time.

Terms	2		3		6		11		18
Difference		1		3		5		7	
Second difference			2		2		2		

The value of a is half the second difference. (For a challenge, try to prove this. Start by writing down expressions in terms of a, b and c for the 1st, 2nd and 3rd terms in the quadratic sequence $an^2 + bn + c$).

So in the example above, the value of a is 1. Now, to find the values of b and c, you can subtract an^2 from each of the terms of the sequence, leaving the sequence $bn + c$.

Terms	2	3	6	11	18
an^2	1	4	9	16	25
$bn + c$	1	-1	-3	-5	-7

The values of $bn + c$ go down by 2 each time. This is a linear sequence, so b must be -2, and you can see that c must be 3 to give the correct terms.

So the quadratic sequence has nth term $n^2 - 2n + 3$.

Example 5
The first five terms of a quadratic sequence are 1, 8, 19, 34, 53.
Find an expression for the nth term.

Solution

The nth term is $an^2 + bn + c$.

Terms	1		8		19		34		53
Differences		7		11		15		19	
Second differences			4		4		4		

So $a = 2$.

Terms	1	8	19	34	53
an^2	2	8	18	32	50
$bn + c$	-1	0	1	2	3

The values of $bn + c$ go up by 1 each time, so $b = 1$

For the first term, $bn + c = -1$, so $1 + c = -1$, so $c = -2$

The nth term of the quadratic sequence is $2n^2 + n - 2$.

Remember that you can check your answer by substituting values for n.

The limit of a sequence

In some sequences (called convergent sequences), the terms of the sequence get closer and closer to a particular number, called the limit of the sequence. You can find a limit by thinking about what happens when the value of n gets very large, as shown in the next example.

Example 6

The nth term of a sequence is given by $\dfrac{4n+1}{2n-3}$

Find the limit of the sequence.

Solution

As $n \to \infty$, $\quad 4n + 1 \to 4n.$

$$2n - 3 \to 2n.$$

$$\frac{4n+1}{2n-3} \to \frac{4n}{2n} = 2$$

So the limit of the sequence is 2.

$\to$ stands for 'tends to' which means 'becomes closer and closer to'. The symbol ∞ represents infinity. So 'as $n \to \infty$' means 'as n tends to infinity'.