Section 2: Further differentiation

Notes and Examples

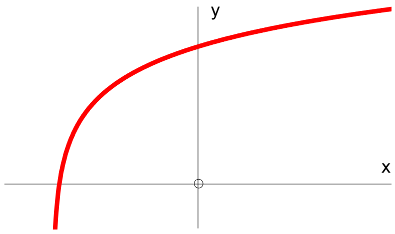
These notes contain sub-sections on:

Increasing and decreasing functions

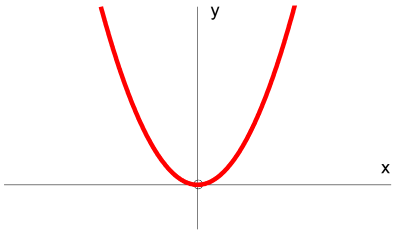
A function is called an increasing function if the value of $\mathrm{f}(x)$ is increasing as x increases. A function can be an increasing function for all values of x, or just for some values of x.

Similarly, a function is called a decreasing function if the value of $\mathrm{f}(x)$ is decreasing as x decreases.

This function is an increasing function for all values of x



This function is a decreasing function for $x < 0$, and an increasing function for $x > 0$.



For an increasing function, the gradient is positive, and for a decreasing function, the gradient is negative.

Turning points

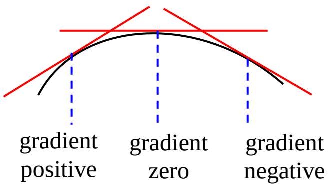
Points on a curve where the tangent is horizontal are called *stationary points.*

At these points, the gradient of the curve is zero, so $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$.

Stationary points are classified into three different types:

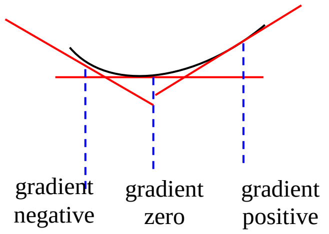
Local **maximum**

The gradient is positive to the left, zero at the point, and negative to the right.



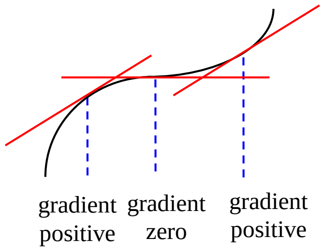
Local **minimum**

The gradient is negative to the left, zero at the point, and positive to the right.

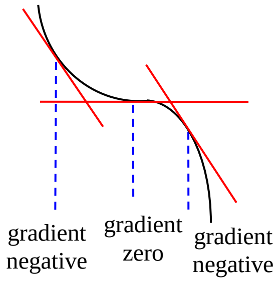


Stationary point of inflection

The gradient goes from positive to zero to positive



or negative to zero to negative.



To distinguish between these, you can test the value of the derivative either side of the stationary point.

Example 1

Find the stationary points on the curve $y = 3x - x^3$, investigate their nature, and sketch the curve.

Solution

Step 1: Differentiate the function.

$$y = 3x - x^3$$

$$\frac{dy}{dx} = 3 - 3x^2.$$

Step 2: Solve $\frac{dy}{dx} = 0$

$$3 - 3x^2 = 0$$

$\Rightarrow \quad 3 = 3x^2$

$\Rightarrow \quad x^2 = 1$

$\Rightarrow \quad x = -1 \text{ or } 1$

When $x = -1$, $y = 3\times(-1) - (-1)^3 = -3 - (-1) = -2$;

When $x = 1$, $y = 3\times1 - 1^3 = 2$.

Step 3: Calculate the y–coordinates for these values of x (called the stationary values).

So the stationary points are $(-1, -2)$ and $(1, 2)$.

Step 4: Use a table to investigate the sign of $\frac{dy}{dx}$ for values of x either side of the stationary values

x	-2	-1	0	1	2
$\frac{dy}{dx}$	-9 -ve	0	3 +ve	0	-9 -ve

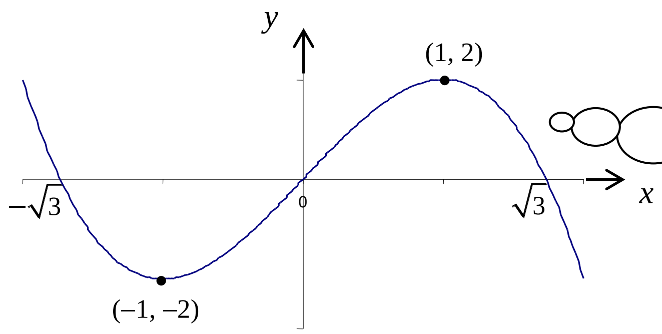
So $(-1, -2)$ is a local minimum and $(1, 2)$ is a local maximum

The curve crosses the $y-$ axis when $x = 0$.
When $x = 0$, $y = 0$.

Step 5: Find where the curve cuts the axes

It crosses the x–axis when $y = 0$.
When $y = 0$, $\quad 3x - x^3 = 0$

$$\Rightarrow x(3 - x^2) = 0,$$
$$\Rightarrow x = 0,\ \sqrt{3} \text{ and } -\sqrt{3}.$$



Step 6: Sketch the curve. Make sure your sketch includes the coordinates of the intercepts and the turning points.

The second derivative

When a function is differentiated and the differential is differentiated again we call this the second derivative of the function. The second derivative shows the gradient of the first derivative.

The second derivative is written as $\frac{d^2y}{dx^2}$ or sometimes $f''(x)$.

Example 2

Find the first and second derivative of the curve $y = 3x^3 - 6x + 5$

Solution

$$y = 3x^3 - 6x + 5$$
$$\frac{dy}{dx} = 6x^2 - 6$$
$$\frac{d^2y}{dx^2} = 12x$$

When you find the second differential we do exactly the same as differentiating any other function. You are just differentiating the first differential

The second derivative can be used to determine the nature of a stationary point.

If $\frac{d^2y}{dx^2}$ is positive at a stationary point, where $\frac{dy}{dx}$=0, then the gradient of the first derivative is positive so the gradient of the original function must go from negative to positive, in which case the turning point must be a minimum.

Conversely, if $\frac{d^2y}{dx^2}$ is positive at a stationary point, then the gradient of the first derivative is negative so the gradient of the original function must go from must go from positive to negative, in which case the turning point must be a maximum.

Example 3

Find the location of the stationary points for the function $y = 2x^3 - 6x^2$ and determine the nature of them

Solution

$$y = 2x^3 - 6x^2$$
$$\frac{dy}{dx} = 6x^2 - 12x$$

- Differentiate the function
- Solve $\frac{dy}{dx} = 0$
- Find the y coordinate

$6x^2 - 12x=0$

$x^2 - 2x=0$

$x(x - 2) = 0$

$x = 2$ or $x = 0$

$y = 2(2)^3 - 6(2)^2 = -8$ $\quad$ $y = 2(0)^3 - 6(0)^2 = 0$

Stationary points are at (2,-8) and (0,0)

Find the second derivative, $\frac{d^2y}{dx^2}$

$$\frac{d^2y}{dx^2} = 12x - 12$$

at x=2, $\frac{d^2y}{dx^2}$=12 so (2,-8) is a minimum.

at x=0, $\frac{d^2y}{dx^2}$=-12 so (0,0) is a maximum.