AQA Level 2 Further Mathematics Algebra IV

Section 1: Solving linear and quadratic equations

Notes and Examples

These notes contain subsections on

Linear equations

A linear equation involves only terms in x (or whatever variable is being used) and numbers. So it has no terms involving x^2, x^3 etc. Equations like these are called linear because the graph of an expression involving only terms in x and numbers (e.g. $y = 2x + 1$) is always a straight line.

Solving a linear equation may involve simple algebraic techniques such as gathering like terms and multiplying out brackets. Example 1 shows a variety of techniques that you might need to use.

Example 1

Solve these equations.

(i) $5x-2=3x+8$

(ii) $3(2y-1)=4-2(y-3)$

(iii) $\dfrac{2a-1}{3}=2a+3$

Solution

(i)

$5x-2=3x+8$

Add 2 to each side

$5x=3x+8+2$

$5x=3x+10$

Subtract $3x$ from each side

$5x-3x=10$

$2x=10$

Divide each side by 2

$x=5$

(ii)

$3(2y-1)=4-2(y-3)$

Multiply out the brackets

$6y-3=4-2y+6$

$6y-3=10-2y$

Add 3 to each side

$6y=13-2y$

Add $2y$ to each side

$8y=13$

Divide each side by 8

$y=\frac{13}{8}$

(iii) $\dfrac{2a-1}{3} = 2a+3$

$2a-1 = 3(2a+3)$ Multiply both sides by 3

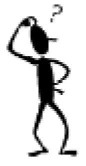
$2a-1 = 6a+9$ Multiply out the brackets

$2a = 6a+10$ Add 1 to each side

$-4a = 10$ Subtract $6a$ from each side

$a = -2.5$ Divide both sides by -4

In Example 2, the problem is given in words and you need to express this algebraically before solving the equation.



Example 2
Jamila has a choice of two tariffs for text messages on her mobile phone.
Tariff A: 10p for the first 5 messages each day, 2p for all others
Tariff B: 4p per message
How many messages would Jamila need to send each day for the two tariffs to cost the same? (She always sends at least 5!)

Solution
Let the number of messages Jamila sends per day be n.
Under Tariff A, she has to pay 10p for each of 5 messages and 2p for each of n - 5 messages.
Cost = $50+2(n-5)$
Under Tariff B, she has to pay 4p for each of n messages.
Cost = $4n$
For the cost to be the same

$$\begin{aligned} 50+2(n-5) &= 4n \\ 50+2n-10 &= 4n \\ 40+2n &= 4n \\ 40 &= 2n \\ 20 &= n \end{aligned}$$

She needs to send 20 messages per day for the two tariffs to cost the same.

Solving quadratic equations by factorisation

Solving quadratic equations is important not just from the algebraic point of view, but because it gives you information about the graph of a quadratic function. The solutions of the equation $ax^2 + bx + c = 0$ tells you where the graph of the function $y = ax^2 + bx + c$ crosses the x-axis, since these are the points where $y = 0$.

Some quadratic equations can be solved by factorising.

Example 3

Solve these quadratic equations by factorising.

(a) $x^2 + 2x - 8 = 0$ (b) $2x^2 + 11x + 12 = 0$

Solution

(a) $x^2 + 2x - 8 = 0$

$(x + 4)(x - 2) = 0$

$x + 4 = 0$ or $x - 2 = 0$

$x = -4$ or 2

For this expression to be zero, one or other of the factors must be zero.

(b) $2x^2 + 11x + 12 = 0$

$(2x + 3)(x + 4) = 0$

$2x + 3 = 0$ or $x + 4 = 0$

$x = -\frac{3}{2}$ or -4

Solving quadratic equations that cannot be factorised

Many quadratic expression cannot be factorised. It is possible to use the technique of completing the square to solve a quadratic equation that cannot be factorised.

Example 4

Solve the equation $x^2 + 4x - 7 = 0$.

Solution

The quadratic expression cannot be factorised.

$$(x+2)^2 = x^2 + 4x + 4$$

$$x^2 + 4x = 7$$

$$(x+2)^2 - 4 = 7$$

$$(x+2)^2 = 11$$

$$x + 2 = \pm\sqrt{11}$$

$$x = -2 \pm \sqrt{11}$$

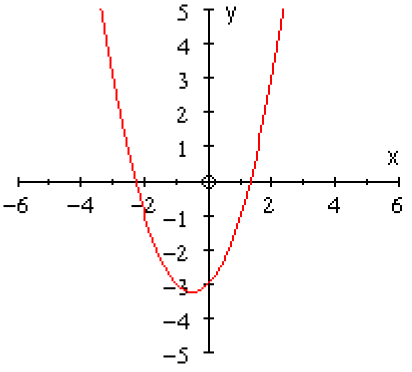
However, completing the square is not usually the best way to solve a quadratic equation which cannot be factorised. Unless your quadratic expression is already in the completed square form, it is usually easier to use the **quadratic formula**.

The quadratic formula for the solutions of the equation $ax^2 + bx + c = 0$ is

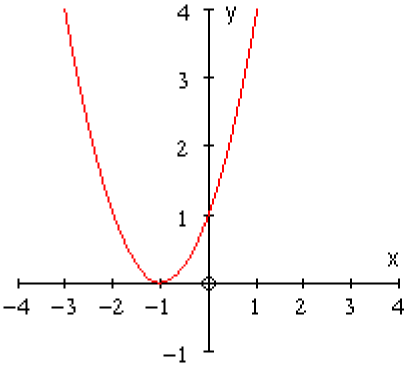
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This formula is derived from completing the square for the equation $ax^2 + bx + c = 0$ - if you would like a challenge, try to prove it using completing the square!

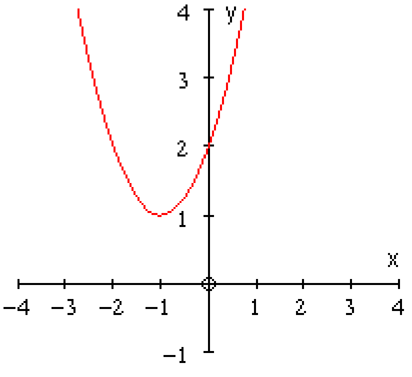
The expression b^2-4ac (called the **discriminant**) is very important as it tells you something about the nature of the roots.



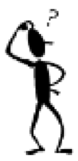
$y = x^2 + x - 3$
$b^2 - 4ac = 13$
Two real roots



$y = x^2 + 2x + 1$
$b^2 - 4ac = 0$
One real root



$y = x^2 + 2x + 2$
$b^2 - 4ac = -4$
No real roots



Example 5
For each of the following quadratic equations, solve the equation, where possible, by a suitable method.
(i) $2x^2-5x+1=0$ (ii) $6x^2+11x-10=0$

Solution
(i) $a = 2$, $b = -5$, $c = 1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{5 \pm \sqrt{17}}{2 \times 2}$$
$$= \frac{5 \pm \sqrt{17}}{4}$$

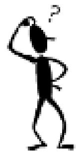
(ii) $a = 3$, $b = -2$, $c = 4$

$b^2 - 4ac = (-2)^2 - 4 \times 3 \times 4 = 4 - 48 = -44$

There are no real solutions.

Problem solving

Some problems, when translated into algebra, involve quadratic equations.



Example 6

A rectangular box has width 2 cm greater than its length, and height 3 cm less than its length.
The total surface area of the box is 548 cm².
What are the dimensions of the box?

Solution

Let the length of the box be x cm.
The width of the box is $x + 2$ cm, and the height is $x - 3$ cm.

The surface are of the box is given by $2x(x + 2) + 2x(x - 3) + 2(x + 2)(x - 3)$

$2x(x + 2) + 2x(x - 3) + 2(x + 2)(x - 3) = 548$

Divide through by 2

$x(x + 2) + x(x - 3) + (x + 2)(x - 3) = 274$

$x^2 + 2x + x^2 - 3x + x^2 - x - 6 = 274$

$3x^2 - 2x - 280 = 0$

The discriminant is 3364, which is 58², so this must factorise

$(3x + 28)(x - 10) = 0$

$x = 10$

$3x + 28 = 0$ gives a negative value of x, which does not make sense in this context. So the solution must be $x - 10 = 0$.

The length of the box is 10 cm, the width is 12 cm and the height is 7 cm.

Notice that in Example 10, you could discard one of the possible solutions as a negative solution did not make sense in the context. This is not always the case. In some situations, a negative solution can have a practical meaning. For example if the height of a stone thrown from the edge of a cliff is negative, this simply means that the stone is below the level of the cliff at that point. However, if the stone was thrown from level ground, then a negative height does not make sense.

Some problems leading to quadratic equations do have two possible solutions.
Always consider whether your solution(s) make sense in the context.