

Section 3: The factor theorem

Notes and Examples

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The factor theorem

You already know that you can solve some quadratics by factorising them.

e.g. to solve the quadratic equation $x^2 + 3x - 10 = 0$
 you factorise: $(x + 5)(x - 2) = 0$
 and deduce the solutions $x = -5$ and $x = 2$

Clearly, for $f(x) = x^2 + 3x - 10$, $f(-5) = 0$ and $f(2) = 0$.

$(x + 5)$ is a factor of $f(x) \Leftrightarrow f(-5) = 0$

$(x - 2)$ is a factor of $f(x) \Leftrightarrow f(2) = 0$

This idea can be extended to other polynomials such as cubics.

For example, for the cubic function $g(x) = (x - 1)(x + 2)(x - 3)$, $g(1) = 0$, $g(-2) = 0$ and $g(3) = 0$.

$(x - 1)$ is a factor of $g(x) \Leftrightarrow g(1) = 0$

$(x + 2)$ is a factor of $g(x) \Leftrightarrow g(-2) = 0$

$(x - 3)$ is a factor of $g(x) \Leftrightarrow g(3) = 0$

In general, the **factor theorem** states that:

If $(x - a)$ is a factor of $f(x)$, then $f(a) = 0$ and $x = a$ is a root of the equation $f(x) = 0$.

Conversely, if $f(a) = 0$, then $(x - a)$ is a factor of $f(x)$.

The factor theorem is useful for factorising cubic expressions and for solving cubic equations.

As with quadratics, it may or may not be possible to factorise a cubic expression. If it is possible, the first step is to factorise it into a linear factor and a quadratic factor. Then it may be possible to factorise the quadratic factor into two further linear factors.

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Example 1

- (i) Show that $x + 2$ is a factor of $x^3 - 2x^2 - 5x + 6$.
- (ii) Show that $x^3 - 2x^2 - 5x + 6 = (x + 2)(x^2 - 4x + 3)$.
- (iii) Hence solve the equation $x^3 - 2x^2 - 5x + 6 = 0$.

Solution

- (i) $f(x) = x^3 - 2x^2 - 5x + 6$
 $f(-2) = (-2)^3 - 2(-2)^2 - 5 \times -2 + 6$
 $= -8 - 8 + 10 + 6$
 $= 0$
 $f(-2) = 0$ so $(x + 2)$ is a factor of $f(x)$.

- (ii) $(x + 2)(x^2 - 4x + 3) = x(x^2 - 4x + 3) + 2(x^2 - 4x + 3)$
 $= x^3 - 4x^2 + 3x + 2x^2 - 8x + 6$
 $= x^3 - 2x^2 - 5x + 6$

- (iii) $x^3 - 2x^2 - 5x + 6 = 0$

$$(x + 2)(x^2 - 4x + 3) = 0$$

$$(x + 2)(x - 1)(x - 3) = 0$$

The roots of the equation are $x = -2$, $x = 1$ and $x = 3$.

The quadratic factor
can be factorised into
two linear factors

In the example above, you were given the factorisation into a linear and a quadratic factor, and you just needed to show that it was correct. More often you will need to find the quadratic factor yourself. So you need a method of dividing a cubic expression by a linear factor to give the quadratic factor.

There are several different methods of doing this.

Dividing a cubic expression by a linear factor

In this course, all the cubic expressions you will see in this course have 1 as the coefficient of x^3 . This means that the quadratic factor will have 1 as the coefficient of x^2 . So you can express the quadratic factor as $x^2 + px + q$, and then multiply out and find the values of p and q by comparing with the original expression.



Example 2

- (i) Show that $x - 1$ is a factor of $x^3 - 3x^2 - x + 3$.
- (ii) Factorise $x^3 - 3x^2 - x + 3$ completely.

Solution

- (i) $f(x) = x^3 - 3x^2 - x + 3$
 $f(1) = 1 - 3 - 1 + 3 = 0$
so $x - 1$ is a factor.



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$$\begin{aligned} \text{(ii)} \quad x^3 - 3x^2 - x + 3 &= (x-1)(x^2 + px + q) \\ &= x(x^2 + px + q) - 1(x^2 + px + q) \\ &= x^3 + px^2 + qx - x^2 - px - q \\ &= x^3 + (p-1)x^2 + (q-p)x - q \end{aligned}$$

$$\text{Equating coefficients of } x^2 \quad \Rightarrow p - 1 = -3 \Rightarrow p = -2$$

$$\text{Equating constant terms} \quad \Rightarrow -q = 3 \Rightarrow q = -3$$

$$(\text{Check: Coefficients of } x = q - p = -3 - (-2) = -1)$$

$$\begin{aligned} x^3 - 3x^2 - x + 3 &= (x-1)(x^2 - 2x - 3) \\ &= (x-1)(x+1)(x-3) \end{aligned}$$

In the example above, you may have noticed that it is very easy to find the value of q , just by thinking about the constant term. It is possible to do the factorisation 'in your head', without writing it out using p and q .

It's also clear that the constant term in the quadratic factor must be -3 , to give the constant term 3 using $-1 \times -3 = 3$.

So the factorisation can be written like this:

$$x^3 - 3x^2 - x + 3 = (x-1)(x^2 + \square x - 3)$$

You can then work out the missing coefficient by thinking about either the terms in x^2 or the terms in x . Multiplying out the linear and quadratic factors will give a term of $-x^2$ (from multiplying -1 in the linear factor by x^2 in the quadratic factor). So we need another $-2x^2$ to get a total of $-3x^2$. So the x term in the quadratic factor must be $-2x$, so that we get $-2x^2$ by multiplying x in the linear factor by $-2x$ in the quadratic factor.

$$x^3 - 3x^2 - x + 3 = (x-1)(x^2 - 2x - 3)$$

Don't worry if you find this method too difficult at first. It's fine to use the method shown in Example 2.



Example 3

$f(x) = x^3 + px^2 + 11x - 6$ has a factor $x - 2$.

Find the value of p and hence factorise $f(x)$ as far as possible.

Solution

$x - 2$ is a factor of $f(x) \Leftrightarrow f(2) = 0$

$$f(2) = 8 + 4p + 22 - 6 = 24 + 4p$$

$$24 + 4p = 0 \Rightarrow p = -6$$

$$f(x) = x^3 - 6x^2 + 11x - 6$$



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$$\begin{aligned}x^3 - 6x^2 + 11x - 6 &= (x-2)(x^2 - 4x + 3) \\ &= (x-2)(x-1)(x-3)\end{aligned}$$

Further example

The factor theorem can be extended to apply to factors of the form $(ax-b)$. If

$(ax-b)$ is a factor of $f(x)$, then $\left(x-\frac{b}{a}\right)$ is a factor of $f(x)$. So by the factor theorem, $f\left(\frac{b}{a}\right)=0$. Conversely, if $f\left(\frac{b}{a}\right)=0$ then $\left(x-\frac{b}{a}\right)$ is a factor of $f(x)$ and so $(ax-b)$ is a factor of $f(x)$.



Example 4

- (i) Show that $(2x-1)$ is a factor of $f(x)=2x^3+3x^2-6x+2$.
- (ii) Hence solve the equation $2x^3+3x^2-6x+2=0$.

Solution

$$\begin{aligned}\text{(i)} \quad f\left(\frac{1}{2}\right) &= 2\left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^2 - 6\left(\frac{1}{2}\right) + 2 \\ &= \frac{1}{4} + \frac{3}{4} - 3 + 2 \\ &= 0\end{aligned}$$

$f\left(\frac{1}{2}\right)=0 \Rightarrow (x-\frac{1}{2})$ is a factor of $f(x)$, so $(2x-1)$ is a factor of $f(x)$.

$$\begin{aligned}\text{(ii)} \quad 2x^3 + 3x^2 - 6x + 2 &= 0 \\ (2x-1)(x^2 + 2x - 2) &= 0\end{aligned}$$

Applying the quadratic formula to $x^2 + 2x - 2 = 0$

$$\text{gives } x = \frac{-2 \pm \sqrt{4 - 4 \times 2 \times -2}}{2} = \frac{-2 \pm \sqrt{20}}{2} = \frac{-2 \pm 2\sqrt{5}}{2} = -1 \pm \sqrt{5}$$

So the roots of the equation are $x = \frac{1}{2}$ and $x = -1 \pm \sqrt{5}$

