

Section 4: The product rule for counting

Notes and Examples

These notes contain subsections on

- [Products, including factorials and tree diagrams](#)

Products, including factorials and tree diagrams

Sometimes it's possible to think of how many possibilities there are for an object or thing by thinking about how many possibilities there are at each stage of its construction and then multiplying these together. The first example shows this.



Example 1

A 3 letter code is made using the letters X, Y and Z. Each letter can be used as many times as you like.

How many codes are there?

Solution

The first letter can be chosen in 3 ways.

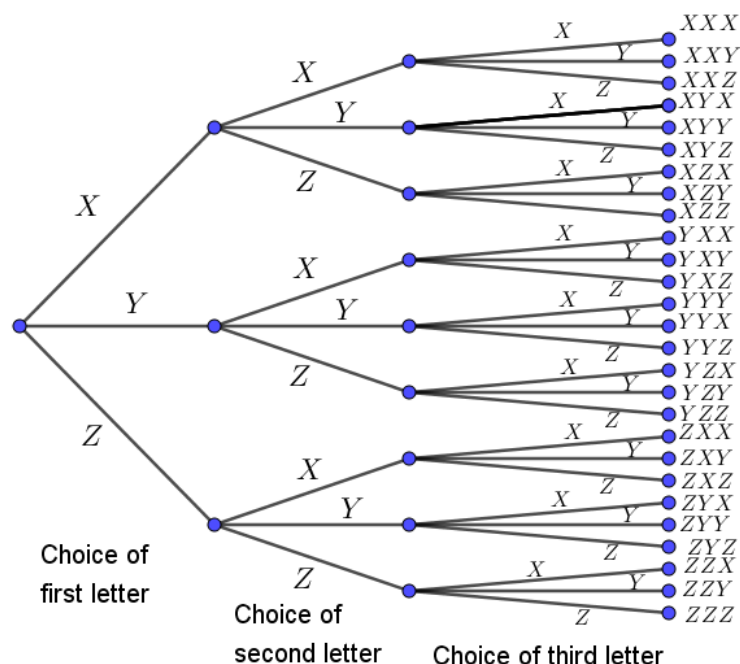
The second letter can also be chosen in 3 ways, as we can repeat the first letter.

The third letter can be chosen in 3 ways.

Altogether we have $3 \times 3 \times 3$ or 3^3 ways.

$$3^3 = 27$$

It's possible to think of the above in terms of a tree diagram as follows



From the left, the first branches show the choice between X, Y or Z for the first letter. The next branches are the choice of the second letter and the branches on the right

are the choice of the third letter. At each stage, each branch splits into three branches. Multiplication can therefore be used to give the total number of branches. You can see that there are $3 \times 3 \times 3 = 27$ distinct paths from the left to the right, each resulting in a different code.

The same principle is applied in the next example. An important aspect of life is setting up a password for entry into a website.



Example 2

A 5 letter password can be made using each of the letters P, Q, R, X, Y and Z as many times as we like.

How many arrangements are there?



Solution

The first letter can be chosen in 6 ways.

The second letter can also be chosen in 6 ways, as we can repeat the first letter.

The third letter can be chosen in 6 ways.

The fourth letter can be chosen in 6 ways.

The fifth letter can be chosen in 6 ways.

Altogether we have $6 \times 6 \times 6 \times 6 \times 6$ or 6^5 ways.

$6^5 = 7776$.

This means the chances of somebody guessing a password made up in this way are $\frac{1}{7776}$.

The next example, where letters cannot be re-used, shows how factorials are involved in counting problems. Note that when you can re-use a letter (or object) it is not a factorial problem (as in the examples above).

Example 3

Two ways of arranging the letters A, B, C, D are ACBD and BDAC (each letter can be used only once). How many different ways are there of arranging the four letters?

Solution

A good way to think of this is that you are going to fill the four boxes below with the four letters.

--	--	--	--

You can put any of the letters in the first box. So there are four different ways of making this first selection. The B in the diagram below could have been any of the four letters.

B			
----------	--	--	--

You can then put any of the three remaining letters in the second box. So there are then three different ways of making this second selection. The D in the diagram below could have been any of the three remaining letters.



Therefore there are $4 \times 3 = 12$ different possibilities for how the boxes could look at this stage (having filled the first two boxes).

Continuing in this way you can see that there will be $4 \times 3 \times 2 \times 1 = 4! = 24$ different ways of arranging the four letters.

The same principle applies in the next example.



Example 4

A 6 letter password can be made using each of the letters P, Q, R, X, Y and Z once. How many arrangements are there?



Solution

The first letter can be chosen in 6 ways.
 Once we have chosen this letter we cannot use it again,
 So the second letter can be chosen in 5 ways.
 The third letter can be chosen in 4 ways.
 The fourth letter can be chosen in 3 ways.
 The fifth letter can be chosen in 2 ways.
 This leaves us with only 1 letter.
 The sixth letter can be chosen in 1 way.

Altogether the 6 letters can be arranged in $6 \times 5 \times 4 \times 3 \times 2 \times 1$ ways, or $6!$ ways and,
 $6! = 720$.

This means the chances of somebody guessing a password made up in this way are

$$\frac{1}{720}$$

Check that you know how to work out factorials on your calculator.
 Some calculators have a $!$ key; in others it will be found on a menu.



Example 5

A password can be made consisting of 3 letters followed by 3 numbers. The first 3 letters are selected using each of the letters A, T and Z once. The numbers are selected from 0, 1 and 2, using each of the numbers once.

How many arrangements are there?



Solution

The first letter can be chosen in 3 ways.

Once we have chosen this letter we cannot use it again,

So the second letter can be chosen in 2 ways.

The third letter can be chosen in 1 way.

The first number can be chosen in 3 ways.

Once we have chosen this number we cannot use it again,

So the second number can be chosen in 2 ways.

The third number can be chosen in 1 way.

Altogether, the 3 letters can be arranged in $3 \times 2 \times 1$ ways = 6 ways.

Altogether, the 3 numbers can be arranged in $3 \times 2 \times 1$ ways = 6 ways.

Altogether, the 3 letters and 3 numbers can be arranged in 6×6 ways = 36 ways.